On the Muskat problem with Surface Tension

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The Muskat Equation with surface tension

Consider two incompressible fluids with different densities and same viscosity in a porous media. The dynamic of the interface between these two fluids is given verifies the following contour integral formulation :

$$(\mathcal{M}_\sigma) : \begin{cases} f_t(x,t) = \dfrac{k}{2\pi\mu} p.v. \displaystyle\int_{\mathbb{R}} \dfrac{y + f_x(x,t)\left(f(x,t) - f(x-y,t)\right)}{y^2 + \left(f(x,t) - f(x-y,t)\right)^2} \\ \qquad\qquad \partial_x \tau_y \left(\sigma \kappa(f) - g\rho f\right)(x,t)\, dy, \\ f(x,0) = f_0(x) \end{cases}$$

$k =$ permeabily of the porous media, σ=surface tension coeff, μ=viscosity, ρ=difference of the densities, g=gravity, κ=curvature of the interface

- This formulation appears in a paper by A-V Matioc and B-V Matioc (Journal of Elliptic and Parabolic Equations, '21)

$$(\mathcal{M}_\sigma): \begin{cases} f_t(x,t) = \dfrac{k}{2\pi\mu} p.v. \displaystyle\int_{\mathbb{R}} \dfrac{y + f_x(x,t)\left(f(x,t) - f(x-y,t)\right)}{y^2 + \left(f(x,t) - f(x-y,t)\right)^2} \\ \qquad\qquad \partial_x \tau_y \left(\sigma\kappa(f) - g\rho f\right)(x,t)\, dy, \\ f(x,0) = f_0(x) \end{cases}$$

Where $\kappa(f) = \kappa(f(x,t))$ is the curvature operator of the moving interface $\{y = f(x,t) + tV\}$ with constant speed V, where $V \in \mathbb{R}$. The curvature operator is defined as follows

$$\kappa(f) := \frac{\partial_{xx} f(x,t)}{(1 + (\partial_x f(x,t))^2)^{\frac{3}{2}}}$$

$$\begin{cases} \partial_t f = \frac{\sigma}{2\pi} \text{ p.v.} \int \frac{1+\partial_x f(x)\Delta_\alpha f(x)}{1+\Delta_\alpha^2 f(x)} \partial_x \left(\frac{\partial_x^2 f(x-\alpha)}{(1+f_x(x-\alpha)^2)^{\frac{3}{2}}} \right) \frac{d\alpha}{\alpha} \\ \qquad + \frac{g\rho}{2\pi} \text{ p.v.} \int \partial_x \arctan \Delta_\alpha f \; d\alpha, \\ f(x,0) = f_0(x). \end{cases}$$

When $\sigma = 0$, the equation is, by now, quite well understood, in particular, the critical regularity issue (in Sobolev) has been already treated in Córdoba-L ('21). In this talk, I will present only the critical Sobolev regularity estimates in the case $g = 0$.

$$\begin{cases} \partial_t f = \frac{\sigma}{2\pi} \text{ p.v.} \int \frac{1+\partial_x f(x)\Delta_\alpha f(x)}{1+\Delta_\alpha^2 f(x)} \partial_x \left(\frac{\partial_x^2 f(x-\alpha)}{(1+f_x(x-\alpha)^2)^{\frac{3}{2}}} \right) \frac{d\alpha}{\alpha} \\ f(x,0) = f_0(x). \end{cases}$$

Critical spaces ?

If f is a solution to the PDE above then, for any $\lambda > 0$, the family of functions

$$f_\lambda(x,t) := \lambda^{-1} f(\lambda x, \lambda^3 t)$$

is also a solution for any initial $\lambda^{-1} f_0(\lambda x)$. Hence, examples of critical spaces are $\mathcal{F}_{1,1}$ (Wiener algebra, $\int |\xi||\widehat{f}(\xi)|d\xi < \infty$), $\dot{W}^{1,\infty}$ and $\dot{H}^{\frac{3}{2}}$

The literature is by now very vast in the case $\sigma = 0$, few of them are

Local existence in any subcritical Sobolev spaces

- B. Pausader, H. Nguyen (ARMA, '20)
- T. Alazard, O.L (ARMA, '20)
- H. Abels, B-V Matioc (Eur. Journal of Applied Math, '23)

Global or Local existence in critical spaces

- P. Constantin, D. Córdoba, F. Gancedo, R. Strain (JEMS, '13)
- P. Constantin, F. Gancedo, R. Shvydkoy, V. Vicol (AIHP, '15)
- F. Lin, L. Zhen (CPAM, '17)
- S. Cameron (A& PDE, '18)
- D. Córdoba, O. L. (AENS '21)
- F. Gancedo and O.L (ARMA, '22)
- T. Alazard, HQ Nguyen (CPDE, APDE, CMP, AIM '21-'22)
- K. Chen, HQ Nguyen, Y. Yu (TAMS '23)

Some authors (and many others) has studied the other desingularizations, mild solutions...

- J. Shi, degenerate regularity near the turnover points (ArXiv)
- E. García-Juárez, J. Gómez-Serrano, HQ. Nguyen and B. Pausader (AIM, '22)
- E. García-Juárez, J. Gómez-Serrano, S. V. Haziot, and B. Pausader (arXiv)
- F. Gancedo, E. García-Juárez, N. Patel, and R. M. Strain, Muskat with viscosity jump (AIM '19) [global large slope is an open problem].

Singularity and lack of uniqueness (stable/instable regime)

- A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and M. Lopez-Fernandez (RT breakdown, Annals of Math, '12)
- A. Castro, D. Córdoba, C. Fefferman, and F. Gancedo (ARMA '13).
- D. Córdoba, J. Gómez-Serrano, and A. Zlatos (A& PDE, PTRSL '15)
- F. Gancedo, R. Strain (PNAS '14)
- A. Ionescu, C. L. Fefferman and V. Lie (Duke Math '16)
- A. Zlatos (sing in the half-plane from arbitrarily smooth small data), Arxiv '24

When $\sigma > 0$ critical regularity is less studied (many open questions regarding singularities).

D. M. Ambrose, The zero surface tension limit of two-dimensional interfacial Darcy flow, J. Math. Fluid Mech., 16 (2014), pp. 105–143.

J. Escher, B.-V. Matioc and C. Walker, The domain of parabolicity for the Muskat problem. Indiana Univ. Math. J. 67, 679–737, 2018.

B.-V. Matioc, Viscous displacement in porous media : the Muskat problem in 2D, Trans. Amer. Math. Soc., 370 (2018), pp. 7511–7556.

H. Q. Nguyen, On well-posedness of the Muskat problem with surface tension, Adv. Math., 374 (2020), p. 107-144.

P. Flynn and Huy Q. Nguyen, The vanishing surface tension limit of the Muskat problem. Communication in Mathematical Physics, Volume 382, pages 1205–1241, (2021)

A-V Matioc and B-V Matioc, A new reformulation of the Muskat problem with surface tension, Journal of Differential Equations (2023)

The main result

Theorem (L, 2024)

There exists a universal constant $C > 0$, such that for any $f_0 \in H^{\frac{5}{2}}$ satisfying

$$\left(C\|f_0\|_{\dot{H}^{\frac{3}{2}}} + C\|f_0\|^4_{\dot{H}^{\frac{3}{2}}} + \|f_0\|_{\dot{H}^{\frac{3}{2}}}\|f_0\|_{\dot{B}^1_{\infty,1}} \right) \left(1 + \|f_0\|^2_{\dot{W}^{1,\infty}}\right)^{\frac{3}{2}} < 1,$$

there exists a unique global solution f to the Muskat problem with surface tension which verifies

$$f \in L^\infty([0,\infty], H^{\frac{5}{2}}) \cap L^2([0,\infty], \dot{H}^4 \cap \dot{H}^3).$$

Recall that,

$$\begin{cases} \partial_t f = \int \frac{1+\partial_x f(x)\Delta_\alpha f(x)}{1+\Delta_\alpha^2 f(x)} \partial_x \left(\frac{\partial_x^2 f(x-\alpha)}{(1+f_x(x-\alpha)^2)^{\frac{3}{2}}} \right) \frac{d\alpha}{\alpha} \\ f(x,0) = f_0(x). \end{cases}$$

Use the following identity,

$$\frac{1}{1+\xi^2} = \int_0^\infty e^{-\delta} \cos(\delta\xi) d\delta$$

Evaluating at $\xi = \Delta_\alpha f$, we find

$$\frac{1}{1+(\Delta_\alpha f)^2} = \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f)\ d\delta.$$

We also have

$$\frac{1}{(1+\eta^2)^{\frac{3}{2}}} = \int_0^\infty e^{-\delta} \cos(\delta\eta)\cos(\arctan \eta) d\delta$$

Evaluating at $\eta = f_x(x-\alpha)$ one finds

$$\frac{1}{(1+f(x-\alpha)^2)^{\frac{3}{2}}} = \int_0^\infty e^{-\delta} \cos(\delta f(x-\alpha))\cos(\arctan f(x-\alpha)) d\delta$$

With this point of view : Easy to symmetrize and, crucially, one may use Besov spaces with regularity $s \in (0,1)$ and finally it is easy to compute derivatives !

With this formulation, we may write

$$\begin{aligned}
\partial_t f &= \frac{1}{\pi} \int \int_0^\infty \int_0^\infty \cos(\gamma \Delta_\alpha f)\ e^{-\gamma-\sigma} \\
&\quad \times \partial_x \left(\frac{\partial_x^2 \tau_\alpha f}{\alpha} \cos(\sigma \tau_\alpha f_x) \cos(\arctan(\tau_\alpha f_x)) \right)\ d\gamma\ d\sigma\ d\alpha \\
&+ \frac{1}{\pi} f_x \int \Delta_\alpha f\ \cos(\gamma \Delta_\alpha f)\ e^{-\gamma-\sigma} \\
&\quad \times \partial_x \left(\frac{\partial_x^2 \tau_\alpha f}{\alpha} \cos(\sigma \tau_\alpha f_x) \cos(\arctan(\tau_\alpha f_x)) \right)\ d\gamma\ d\sigma\ d\alpha
\end{aligned}$$

Then, we linearize the first term in order to extract some weighted parabolic term with nice properties plus a controlled term, more precisely, one uses the decomposition $\cos(\gamma\Delta_\alpha f) = 1 - 2\sin^2(\frac{\gamma}{2}\Delta_\alpha f)$ to get that

$$\begin{aligned}
\partial_t f &= \underbrace{\frac{1}{\pi}\int\int_0^\infty e^{-\sigma}\partial_x\left(\frac{\partial_x^2\tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\right)\, d\sigma d\alpha}_{\text{Hibert transform}} \\
&- \frac{2}{\pi}\int\int_0^\infty\int_0^\infty \sin^2(\frac{1}{2}\Delta_\alpha(\gamma f)(x))\; e^{-\gamma-\sigma} \\
&\quad \times \partial_x\left(\frac{\partial_x^2\tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\right)\, d\gamma\; d\sigma\; d\alpha \\
&+ \frac{1}{\pi}\int\int_0^\infty\int_0^\infty f_x\Delta_\alpha f(x)\;\cos(\gamma\Delta_\alpha f)\; e^{-\gamma-\sigma} \\
&\quad \times \partial_x\left(\frac{\partial_x^2\tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\right)\, d\gamma\; d\sigma\; d\alpha
\end{aligned}$$

$$
\begin{aligned}
\partial_t f &= -\Lambda^3 f \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\; d\sigma \\
&+ \partial_x \left[\mathcal{H}, \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\; d\sigma \right] f_{xx} \\
&+ \mathcal{H} f_{xx}\; \partial_x \left(\int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\; d\sigma \right) \\
&- \frac{2}{\pi} \int \int_0^\infty \int_0^\infty e^{-\gamma-\sigma} \sin^2(\frac{\gamma}{2} \Delta_\alpha f) G_\sigma(f)\; d\alpha\; d\gamma\; d\sigma \\
&+ \frac{1}{\pi} \int \int_0^\infty \int_0^\infty e^{-\gamma-\sigma}\; f_x\; \sin(\gamma \Delta_\alpha f)\; G_\sigma(f)\; d\alpha\; d\gamma\; d\sigma.
\end{aligned}
$$

where $G_\sigma(f) := \partial_x \left(\frac{\partial_x^2 \tau_\alpha f}{\alpha} \cos(\sigma \tau_\alpha f_x) \cos(\arctan(\tau_\alpha f_x)) \right)$

We need to get ride of f_x, we use the identity :

$$-2\int_0^\infty e^{-\gamma}\sin^2(\frac{\gamma}{2}\Delta_\alpha(f)(x))\ d\gamma = -\Delta_\alpha f\int_0^\infty e^{-\gamma}\sin(\gamma\Delta_\alpha f)\ d\gamma.$$

Then,

$$\begin{aligned} - & \frac{2}{\pi}\int\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\sin^2(\frac{\gamma}{2}\Delta_\alpha f)G_\sigma(f)\ d\alpha\ d\gamma\ d\sigma \\ + & \frac{1}{\pi}\int\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\ f_x\ \sin(\gamma\Delta_\alpha f)G_\sigma(f)\ d\alpha\ d\gamma\ d\sigma. \\ = & \frac{1}{\pi}\int\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\ (f_x-\Delta_\alpha f)\sin(\gamma\Delta_\alpha f)G_\sigma(f)\ d\alpha\ d\gamma\ d\sigma. \end{aligned}$$

$$
\begin{aligned}
\partial_t f &= -\Lambda^3 f \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\ d\sigma \\
&+ \partial_x \left[\mathcal{H}, \int_0^\infty \int_0^\infty e^{-\gamma-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\ d\gamma\ d\sigma \right] f_{xx} \\
&+ \mathcal{H} f_{xx}\ \partial_x \left(\int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\ d\sigma \right) \\
&+ \frac{1}{\pi} \int \int_0^\infty \int_0^\infty e^{-\gamma-\sigma}\ (f_x - \Delta_\alpha f) \sin(\gamma \Delta_\alpha f) G_\sigma(f)\ d\alpha\ d\gamma\ d\sigma.
\end{aligned}
$$

where $G_\sigma(f) := \partial_x \left(\frac{\partial_x^2 \tau_\alpha f}{\alpha} \cos(\sigma \tau_\alpha f_x) \cos(\arctan(\tau_\alpha f_x)) \right)$

$f_x - \Delta_\alpha f$ is nicer than f_x, indeed, we have a nice cancellation

$$
\begin{aligned}
f_x - \Delta_\alpha f &= f_x - \frac{1}{2} S_\alpha f - \frac{1}{2} D_\alpha f \\
&= \not{f_x} - \frac{1}{2} S_\alpha f - \frac{1}{2\alpha} \int_0^\alpha s_\eta f_x \, d\eta - \not{f_x}
\end{aligned}
$$

- Note that $\frac{1}{2\alpha} \int_0^\alpha s_\eta f_x \, d\eta$ is a very nice operator (help to milder the strong decay in α).
- As well, $S_\alpha f$ is a regular term as it allows us to use the homogeneous Besov spaces with $s \in [1, 2)$.

$$
\begin{aligned}
\partial_t f \;=\;& -\Lambda^3 f \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\; d\sigma \\
&+ \partial_x \left[\mathcal{H}, \int_0^\infty \int_0^\infty e^{-\gamma-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\; d\gamma\; d\sigma \right] f_{xx} \\
&+ \mathcal{H} f_{xx}\; \partial_x \left(\int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\; d\sigma \right) \\
&- \frac{1}{2\pi} \int \int_0^\infty \int_0^\infty e^{-\gamma-\sigma}\; S_\alpha f\; \sin(\gamma \Delta_\alpha f) G_\sigma(f)\; d\alpha\; d\gamma\; d\sigma \\
&- \frac{1}{2\pi} \int \int_0^\infty \int_0^\infty e^{-\gamma-\sigma} \int_0^\alpha s_\eta f_x\; \sin(\gamma \Delta_\alpha f) G_\sigma(f)\; d\eta\; d\alpha\; d\gamma\; d\sigma
\end{aligned}
$$

where $G_\sigma(f) := \partial_x \left(\frac{\partial_x^2 \tau_\alpha f}{\alpha} \cos(\sigma \tau_\alpha f_x) \cos(\arctan(\tau_\alpha f_x)) \right)$

Oscillatory integrals are much more flexible than rational functions of the slopes. Indeed,

$$\frac{1}{1+(\Delta_\alpha f)^2} = \frac{1}{1+(\Delta_\alpha f+\bar{\Delta}_\alpha f)^2 - 2\Delta_\alpha f\bar{\Delta}_\alpha f - (\bar{\Delta}_\alpha f)^2}$$

However, with the oscillatory integrals $\int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f)d\delta$, we write

$$\frac{1}{2}\int_0^\infty e^{-\delta}(\cos(\delta\Delta_\alpha f)+\cos(\delta\bar{\Delta}_\alpha f))d\delta$$
$$+\frac{1}{2}\int_0^\infty e^{-\delta}(\cos(\delta\Delta_\alpha f)-\cos(\delta\bar{\Delta}_\alpha f))d\delta$$

then use : $\cos(a)-\cos(b) = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$...

Reason : Homogeneous Besov spaces when $s \in [1, 2)$,

$$\|f\|_{\dot{B}^s_{p,r}} = \left\| \frac{\|2f(x) - f(x-\alpha) - f(x+\alpha)\|_{L^p}}{|\alpha|^s} \right\|_{L^r(\mathbb{R}, |\alpha|^{-1}dx)} < \infty,$$

For all $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$ which are such that $p_1 \leq p_2$ and $r_1 \leq r_2$ and for all $s \in \mathbb{R}$, we have the following continuous embedding

$$\dot{B}^s_{p_1,r_1}(\mathbb{R}) \hookrightarrow \dot{B}^{s-(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}),$$

when the third indice is 1, we shall use the following real interpolation inequality. There exists a constant $C > 0$, such that for all $(s_1, s_2) \in \mathbb{R}^2$ and all $\theta \in (0, 1)$

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}$$

for all $p \in [1, \infty]$.

We study the time evolution of the critical semi-norm $\dot{H}^{3/2}$.

$$\frac{1}{2}\partial_t \|f\|^2_{\dot{H}^{3/2}} = \int \Lambda^{3/2} f \ \Lambda^{3/2} \partial_t f \ dx = \int \Lambda^{3} f \ \partial_t f \ dx.$$

what is $\partial_t f$?

$$\begin{cases} \partial_t f = & \underbrace{\partial_x \left[\mathcal{H}, \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\ d\sigma\right] \partial_{xx} f}_{\text{commutator}} \\ & - \underbrace{\Lambda^3 f \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\ d\sigma}_{\text{weighted dissipation}} \\ & + \mathcal{H} f_{xx}\ \partial_x \left(\int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x))\ d\sigma \right) \\ & - \int_E \int_0^\alpha e^{-\gamma-\sigma}\ \frac{1}{\alpha} s_\eta f_x \left(\sin(\gamma \Delta_\alpha f) - \sin(\gamma \bar{\Delta}_\alpha f)\right) G_\sigma(f)\ d\eta\ dZ \\ & - \int_E \int_0^\alpha e^{-\gamma-\sigma}\ \frac{1}{\alpha} s_\eta f_x \left(\sin(\gamma \Delta_\alpha f) + \sin(\gamma \bar{\Delta}_\alpha f)\right) G_\sigma(f)\ d\eta\ dZ \\ & - \int_E e^{-\gamma-\sigma}\ S_\alpha f \left(\sin(\gamma \Delta_\alpha f) - \sin(\gamma \bar{\Delta}_\alpha f)\right) G_\sigma(f)\ dZ \\ & - \int_E e^{-\gamma-\sigma}\ S_\alpha f \left(\sin(\gamma \Delta_\alpha f) + \sin(\gamma \bar{\Delta}_\alpha f)\right) G_\sigma(f)\ dZ \end{cases}$$

where $E = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ and $dZ = \frac{1}{4\pi} d\alpha\ d\gamma\ d\sigma$,

$G_\sigma(f) := \partial_x \left(\frac{\partial_x^2 \tau_\alpha f}{\alpha} \cos(\sigma \tau_\alpha f_x) \cos(\arctan(\tau_\alpha f_x)) \right)$

With this formulation we find

$$\begin{aligned}\frac{1}{2}\partial_t \|f\|^2_{\dot{H}^{3/2}} &= -\int |\Lambda^3 f|^2 \; R(f)\; dx \\ &+ \int \Lambda^3 f \; \partial_x \left[\mathcal{H}, R(f)\right] f_{xx} \; dx \\ &+ \int \Lambda^3 f \; \mathcal{H} f_{xx} \; \partial_x R(f) \; dx \\ &+ T_1 \\ &+ \ldots \\ &+ T_6\end{aligned}$$

Where $R(f) = \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x)) \; d\sigma = \frac{1}{(1+(f_x)^2)^{\frac{3}{2}}}$

$$T_c := \int \Lambda^3 f \ \partial_x \left[\mathcal{H}, \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x)) \ d\sigma \right] f_{xx} \ dx.$$

Then, we see that

$$T_c \leq \|f\|_{\dot{H}^3} \left\| \partial_x \left[\mathcal{H}, \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x)) \ d\sigma \right] f_{xx} \right\|_{L^2}$$

Using a commutator estimate

$$T_c \leq \|f\|_{\dot{H}^3} \|f\|_{\dot{H}^2} \left\| \partial_x \left(\int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x)) \ d\sigma \right) \right\|_{BMO}$$

Using the fact that $\dot{H}^{1/2} \hookrightarrow BMO$, we find

$$\begin{aligned} T_c &\leq \|f\|_{\dot{H}^3}\|f\|_{\dot{H}^2}\int_0^\infty e^{-\sigma}\left\|\partial_x\left(\cos(\sigma f_x)\cos(\arctan(f_x))\right)\right\|_{\dot{H}^{\frac{1}{2}}}\; d\sigma \\ &\leq \|f\|_{\dot{H}^3}\|f\|_{\dot{H}^2}\int_0^\infty \sigma e^{-\sigma}\left\|f_{xx}\sin(\sigma f_x)\cos(\arctan(f_x))\right\|_{\dot{H}^{\frac{1}{2}}}\; d\sigma \\ &+ \|f\|_{\dot{H}^3}\|f\|_{\dot{H}^2}\int_0^\infty e^{-\sigma}\left\|\frac{f_{xx}}{1+(f_x)^2}\cos(\sigma f_x)\sin(\arctan(f_x))\right\|_{\dot{H}^{\frac{1}{2}}} d\sigma \end{aligned}$$

We need to estimate the L^2-norm of

$$\mathcal{N}_1 := \Lambda^{\frac{1}{2}}\left(f_{xx}\sin(\sigma f_x)\cos(\arctan(f_x))\right)$$

$$\begin{aligned}\|\mathcal{N}_1\|_{L^2} &\lesssim \|f\|_{\dot{H}^{\frac{5}{2}}} + \|f_{xx}\|_{L^4} \left\|\Lambda^{\frac{1}{2}}\left(\sin(\sigma f_x)\cos(\arctan(f_x))\right)\right\|_{L^4} \\ &+ \int \frac{\|\delta_y f_{xx}\|_{L^4} \ \|\delta_y(\sin(\sigma f_x)\cos(\arctan(f_x)))\|_{L^4}}{|y|^{\frac{3}{2}}} \, dy\end{aligned}$$

Since the function $\psi_\sigma(\cdot) := \sin(\sigma\cdot)\cos(\arctan(\cdot))$ is σ-Lipschitz, we have, by Sobolev embedding and the fact that $\dot{H}^{3/4}$ is stable by composition with any outer Lipschitz function (Bourdau-Meyer), one gets

$$\left\|\Lambda^{\frac{1}{2}}(\psi_\sigma(f_x))\right\|_{L^4} \lesssim \|\sigma f_x\|_{\dot{H}^{3/4}} \lesssim \sigma \|f\|_{\dot{H}^{\frac{7}{4}}}$$

and

$$\|\delta_y(\sin(\sigma f_x)\cos(\arctan(f_x)))\|_{L^4} \lesssim \|\delta_y \psi_\sigma(f_x)\|_{L^4} \lesssim \sigma \|\delta_y f_x\|_{L^4}$$

Hence, using the two above inequalities, we infer that

$$\|\mathcal{N}_1\|_{L^2} \lesssim \|f\|_{\dot{H}^{\frac{5}{2}}} + \|f\|_{\dot{H}^{\frac{9}{4}}} \|f\|_{\dot{H}^{\frac{7}{4}}} + \sigma \int \frac{\|\delta_y f_{xx}\|_{L^4} \ \|\delta_y f_x\|_{L^4}}{|y|^{\frac{3}{2}}} \, dy$$

Therefore,

$$\begin{aligned}\|\mathcal{N}_1\|_{L^2} &\lesssim \|f\|_{\dot{H}^{\frac{5}{2}}} + \|f\|_{\dot{H}^{\frac{9}{4}}}\|f\|_{\dot{H}^{\frac{7}{4}}} \\ &\quad +\sigma\left(\int \frac{\|\delta_y f_{xx}\|_{L^4}^2}{|y|^{\frac{3}{2}}}\, dy\right)^{\frac{1}{2}}\left(\int \frac{\|\delta_y f_{x}\|_{L^4}^2}{|y|^{\frac{3}{2}}}\, dy\right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{\dot{H}^{\frac{5}{2}}} + \|f\|_{\dot{H}^{\frac{9}{4}}}\|f\|_{\dot{H}^{\frac{7}{4}}} + \sigma\|f\|_{\dot{B}^{\frac{9}{4}}_{4,2}}\|f\|_{\dot{B}^{\frac{5}{4}}_{4,2}}\end{aligned}$$

Since $\dot{H}^{\frac{5}{2}} \hookrightarrow \dot{B}^{\frac{9}{4}}_{4,2}$ and $\dot{H}^{\frac{3}{2}} \hookrightarrow \dot{B}^{\frac{5}{4}}_{4,2}$, we find that,

$$\|\mathcal{N}_1\|_{L^2} \lesssim \|f\|_{\dot{H}^{\frac{5}{2}}} + \|f\|_{\dot{H}^{\frac{9}{4}}}\|f\|_{\dot{H}^{\frac{7}{4}}} + \sigma\|f\|_{\dot{H}^{\frac{5}{2}}}\|f\|_{\dot{H}^{\frac{3}{2}}}$$

By interpolation $(H^3 - H^{\frac{3}{2}})$

$$T_c \lesssim \|f\|^2_{\dot{H}^3}\left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|^2_{\dot{H}^{\frac{3}{2}}}\right)$$

For the term,

$$T_{rem} = \int \Lambda^3 f \ \mathcal{H} f_{xx} \ \partial_x R(f) \ dx,$$

where $R(f) = \int_0^\infty e^{-\sigma} \cos(\sigma f_x) \cos(\arctan(f_x)) \ d\sigma$. We notice that

$$\begin{aligned} T_{rem} &\lesssim \|\Lambda^3 f\|_{L^2} \|\mathcal{H} f_{xx}\|_{L^4} \int e^{-\sigma} \|\partial_x \left(\cos(\sigma f_x) \cos(\arctan(f_x))\right)\|_{L^4} \ d\sigma \\ &\lesssim \|\Lambda^3 f\|_{L^2} \|\mathcal{H} f_{xx}\|_{L^4} \int_0^\infty (1+\sigma) e^{-\sigma} \|f_{xx}\|_{L^4} \ d\sigma. \end{aligned}$$

Using the Sobolev embedding $\dot{H}^{\frac{1}{4}} \hookrightarrow L^4$, we find that

$$T_{rem} \lesssim \|f\|_{\dot{H}^3} \|f\|^2_{\dot{H}^{\frac{9}{4}}}$$

Finally, since $\|f\|_{\dot{H}^{\frac{9}{4}}} \leq \|f\|^{\frac{1}{2}}_{\dot{H}^3} \|f\|^{\frac{1}{2}}_{\dot{H}^{\frac{3}{2}}}$, we conclude that

$$T_{rem} \lesssim \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{\frac{3}{2}}}$$

Recall that,

$$
\begin{aligned}
T_3 \quad = \quad & -\frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty \; e^{-\gamma-\sigma}\;\Lambda^3 f\;\; S_\alpha f\left(\sin(\gamma\Delta_\alpha f)+\sin(\gamma\bar{\Delta}_\alpha f)\right) \\
& \times\; \partial_x\left(\frac{\partial_x^2\tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\right)\; d\gamma\; d\sigma\; d\alpha\; dx
\end{aligned}
$$

By using the following straightforward inequalities for any $\sigma \geq 0$, $\gamma \geq 0$:

$$
\left|\sin(\gamma\Delta_\alpha f)+\sin(\gamma\bar{\Delta}_\alpha f)\right| \leq \gamma|S_\alpha f|,
$$

and

$$
|\partial_x \cos(\sigma\tau_\alpha f_x)| \leq \sigma|\tau_\alpha f_{xx}|,
$$

as well as

$$
|\partial_x \cos(\arctan(\tau_\alpha f_x))| \leq |\tau_\alpha f_{xx}|,
$$

One may write that

$$T_3 \lesssim \iint \int_0^\infty \int_0^\infty p_{\gamma,\sigma}\ e^{-\gamma-\sigma}\ |\Lambda^3 f|\ |S_\alpha f|^2 \left|\frac{\partial_x^3 \tau_\alpha f}{\alpha}\right|\ d\gamma\ d\sigma\ d\alpha\ dx$$
$$+\iint \int_0^\infty \int_0^\infty\ p_{\gamma,\sigma}\ e^{-\gamma-\sigma}\ |\Lambda^3 f|\ |S_\alpha f|^2 \left|\frac{\partial_x^2 \tau_\alpha f}{\alpha}\right|\ |f_{xx}|\ d\gamma\ d\sigma\ d\alpha\ dx$$

where $p_{\gamma,\sigma} = (\gamma(1+\sigma)$ Therefore, by using Sobolev interpolation together with the embedding $\dot{H}^{3/2} \hookrightarrow \dot{B}^1_{\infty,2}$, we get

$$\begin{aligned}
T_3 &\lesssim \|\partial_x^3 f\|^2_{L^2} \int \frac{\|s_\alpha f\|^2_{L^\infty}}{|\alpha|^3}\ d\alpha + \|\Lambda^3 f\|_{L^2}\|f_{xx}\|^2_{L^4} \int \frac{\|s_\alpha f\|^2_{L^\infty}}{|\alpha|^3}\ d\alpha \\
&\lesssim (\|f\|^2_{\dot{H}^3} + \|f\|_{\dot{H}^3}\|f\|^2_{\dot{H}^{9/4}})\ \|f\|^2_{\dot{B}^1_{\infty,2}} \\
&\lesssim (\|f\|^2_{\dot{H}^3} + \|f\|^2_{\dot{H}^3}\|f\|_{\dot{H}^{3/2}})\ \|f\|^2_{\dot{H}^{3/2}} \\
&\lesssim \|f\|^2_{\dot{H}^3} \left(\|f\|^2_{\dot{H}^{3/2}} + \|f\|^3_{\dot{H}^{3/2}}\right)
\end{aligned}$$

The most difficult term is T_4 which require to see a cancellation.

$$\begin{aligned} T_4 &= -\frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\Lambda^3 f \\ &\times \left(\int_0^\alpha \frac{1}{\alpha} s_\eta f_x \, d\eta\right)\left(\sin(\gamma\Delta_\alpha f) - \sin(\gamma\bar{\Delta}_\alpha f)\right) \\ &\times \partial_x\left(\frac{\partial_x^2 \tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\right)\, d\gamma\, d\sigma\, d\alpha\, dx \end{aligned}$$

Indeed, we can show that T_4 can be rewritten as follows

$$
\begin{aligned}
T_4 \quad &= \quad -\frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty\int_0^\alpha \; e^{-\gamma-\sigma}\;\Lambda^3 f\;\frac{1}{\alpha}s_\eta f_x\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\\
&\qquad \times\;\frac{\partial_x^3\tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\;d\eta\;d\gamma\;d\sigma\;d\alpha\;dx\\
&+ \quad \frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty\int_0^\alpha \; \sigma e^{-\gamma-\sigma}\;\Lambda^3 f\;\frac{1}{\alpha}s_\eta f_x\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\\
&\qquad \times\;\frac{\partial_x^2\tau_\alpha f}{\alpha}\tau_\alpha f_{xx}\;\sin(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\;d\eta\;d\gamma\;d\sigma\;d\alpha\;dx\\
&+ \quad \frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty\int_0^\alpha \; e^{-\gamma-\sigma}\;\Lambda^3 f\;\frac{1}{\alpha}s_\eta f_x\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\\
&\qquad \times\;\frac{\partial_x^2\tau_\alpha f}{\alpha}\frac{\tau_\alpha f_{xx}}{1+(\tau_\alpha f_x)^2}\cos(\sigma\tau_\alpha f_x)\sin(\arctan(\tau_\alpha f_x))\;d\eta\;d\gamma\;d\sigma\;d\alpha\;dx\\
&:= \quad A+B+C.
\end{aligned}
$$

Then, write $\partial_x^3\tau_\alpha f = -\partial_\alpha\partial_x^2\tau_\alpha f$, then integrate by parts in α,

$$
\begin{aligned}
A &= \frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty\int_0^\alpha e^{-\gamma-\sigma}\,\Lambda^3 f\,\frac{1}{\alpha}s_\eta f_x\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\\
&\quad\times\frac{\partial_\alpha\partial_x^2\tau_\alpha f}{\alpha}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\,d\eta\,d\gamma\,d\sigma\,d\alpha\,dx\\
&= -\frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty\int_0^\alpha \sigma e^{-\gamma-\sigma}\,\Lambda^3 f\,\frac{1}{\alpha}s_\eta f_x\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\\
&\quad\times\frac{\partial_x^2\tau_\alpha f}{\alpha}\tau_\alpha f_{xx}\,\sin(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\,d\eta\,d\gamma\,d\sigma\,d\alpha\,dx\\
&- \frac{1}{4\pi}\int\int\int_0^\infty\int_0^\infty\int_0^\alpha e^{-\gamma-\sigma}\,\Lambda^3 f\,\frac{1}{\alpha}s_\eta f_x\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\\
&\quad\times\frac{\partial_x^2\tau_\alpha f}{\alpha}\frac{\tau_\alpha f_{xx}}{1+(\tau_\alpha f_x)^2}\cos(\sigma\tau_\alpha f_x)\sin(\arctan(\tau_\alpha f_x))\,d\eta\,d\gamma\,d\sigma\,d\alpha\,dx\\
&- \frac{1}{4\pi}\iint\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\,\Lambda^3 f\,\partial_\alpha\left[\frac{1}{\alpha^2}\left(\sin(\gamma\Delta_\alpha f)-\sin(\gamma\bar{\Delta}_\alpha f)\right)\right.\\
&\quad\left.\int_0^\alpha s_\eta f_x\,d\eta\right]\underbrace{\partial_x^2\sigma\tau_\alpha f}_{=-\partial_\alpha\delta_\alpha f_x}\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\,d\gamma\,d\sigma\,d\alpha\,dx
\end{aligned}
$$

$$
\begin{aligned}
T_4 \;=\; & -\frac{1}{2\pi}\int\int\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\;\Lambda^3 f \\
& \partial_\alpha^2\left[\frac{1}{\alpha}\sin(\frac{\gamma}{2}D_\alpha f)\cos(\frac{\gamma}{2}S_\alpha f)\frac{1}{\alpha}\int_0^\alpha s_\eta f_x\;d\eta\right] \\
& \times\;\delta_\alpha f_x\cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\;d\gamma\;d\sigma\;d\alpha\;dx \\
- \; & \frac{1}{2\pi}\int\int\int_0^\infty\int_0^\infty \sigma e^{-\gamma-\sigma}\;\Lambda^3 f \\
& \partial_\alpha\left[\frac{1}{\alpha}\sin(\frac{\gamma}{2}D_\alpha f)\cos(\frac{\gamma}{2}S_\alpha f)\frac{1}{\alpha}\int_0^\alpha s_\eta f_x\;d\eta\right] \\
& \times\;\delta_\alpha f_x\,(\partial_\alpha\tau_\alpha f_x)\sin(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\;d\gamma\;d\sigma\;d\alpha\;dx \\
- \; & \frac{1}{2\pi}\int\int\int_0^\infty\int_0^\infty \sigma e^{-\gamma-\sigma}\;\Lambda^3 f \\
& \times\partial_\alpha\left[\frac{1}{\alpha}\sin(\frac{\gamma}{2}D_\alpha f)\cos(\frac{\gamma}{2}S_\alpha f)\frac{1}{\alpha}\int_0^\alpha s_\eta f_x\;d\eta\right] \\
& \times\;\delta_\alpha f_x\frac{\partial_\alpha\tau_\alpha f_x}{1+(\tau_\alpha f_x)^2}\sin(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\;d\gamma\;d\sigma\;d\alpha\;dx
\end{aligned}
$$

Importantly, derivatives in α does not destroy the symmetry.

Lemma

The following identities hold,

$$
\begin{aligned}
\partial_\alpha S_\alpha f &= \bar{\Delta}_\alpha f_x - \Delta_\alpha f_x - \frac{s_\alpha f}{\alpha^2}\\
\partial_\alpha^2 S_\alpha f &= \frac{s_\alpha f_{xx}}{\alpha} - \frac{\int_0^\alpha s_\kappa f_{xx}(x)\ d\kappa}{\alpha^2} + 2\frac{s_\alpha f}{\alpha^3}\\
\partial_\alpha D_\alpha f &= -\frac{s_\alpha f_x}{\alpha} - \frac{1}{\alpha^2}\int_0^\alpha s_\kappa f_x\ d\kappa\\
\partial_\alpha^2 D_\alpha f &= \frac{d_\alpha f_{xx}}{\alpha} + 2\frac{s_\alpha f_x}{\alpha^2} + \frac{2}{\alpha^3}\int_0^\alpha s_\eta f_x\ d\eta
\end{aligned}
$$

The proof contains some important cancellations.

$$
\begin{aligned}
S_\alpha f &= \frac{f(x)-f(x-\alpha)}{\alpha} + \frac{f(x)-f(x+\alpha)}{\alpha} \\
&= \frac{2f(x)-f(x-\alpha)-f(x+\alpha)}{\alpha}
\end{aligned}
$$

We infer that,

$$
\begin{aligned}
\partial_\alpha S_\alpha f &= \frac{f_x(x-\alpha)-f_x(x+\alpha)}{\alpha} + \left(\frac{f(x-\alpha)+f(x+\alpha)-2f(x)}{\alpha^2}\right) \\
&= \bar{\Delta}_\alpha f_x - \Delta_\alpha f_x - \frac{s_\alpha f}{\alpha^2}
\end{aligned}
$$

Next, we want to get a satisfactory expression *i.e.* involving symmetric terms for higher derivatives, we find

$$\partial_\alpha^2 S_\alpha f = -\left(\frac{f_{xx}(x-\alpha)+f_{xx}(x+\alpha)}{\alpha}\right) + 2\left(\frac{f_x(x+\alpha)-f_x(x-\alpha)}{\alpha^2}\right) + 2\left(\frac{2f(x)-f(x-\alpha)-f(x+\alpha)}{\alpha^3}\right)$$

Since, on one hand,

$$-\left(\frac{f_{xx}(x-\alpha)+f_{xx}(x+\alpha)}{\alpha}\right) = \left(\frac{-f_{xx}(x-\alpha)-f_{xx}(x+\alpha)+2f_{xx}(x)}{\alpha}\right) - \cancel{\frac{2f_{xx}(x)}{\alpha}}$$

and on the other hand,

$$\left(\frac{f_x(x+\alpha)-f_x(x-\alpha)}{\alpha^2}\right) = \left(\frac{\int_0^\alpha f_{xx}(x+\kappa)+f_{xx}(x-\kappa)-2f_{xx}(x)\ d\kappa}{\alpha^2}\right) + \cancel{\frac{2f_{xx}(x)}{\alpha}}$$

Therefore, we find

$$\partial_\alpha^2 S_\alpha f = \frac{s_\alpha f_{xx}}{\alpha} - \frac{\int_0^\alpha s_\kappa f_{xx}(x)\ d\kappa}{\alpha^2} + 2\frac{s_\alpha f}{\alpha^3}$$

This helps to estimate some of the terms of T_4

$$\begin{aligned} T_4 &= -\frac{1}{2\pi}\int\int\int_0^\infty\int_0^\infty\ e^{-\gamma-\sigma}\ \Lambda^3 f \\ &\quad \partial_\alpha^2\left[\frac{1}{\alpha}\sin(\frac{\gamma}{2}D_\alpha f)\cos(\frac{\gamma}{2}S_\alpha f)\frac{1}{\alpha}\int_0^\alpha s_\eta f_x\ d\eta\right] \\ &\quad \times\ \delta_\alpha f_x \cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\ d\gamma\ d\sigma\ d\alpha\ dx \end{aligned}$$

For instance,

$$
\begin{aligned}
T_{4,1,1} &= -\frac{1}{\pi}\int\int\int_0^\infty\int_0^\infty e^{-\gamma-\sigma}\ \Lambda^3 f\ \frac{1}{\alpha^3} \\
&\quad \times \sin(\frac{\gamma}{2}D_\alpha f)\cos(\frac{\gamma}{2}S_\alpha f)\frac{1}{\alpha}\int_0^\alpha s_\eta f_x\ d\eta \\
&\quad \times\ \delta_\alpha f_x \cos(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\ d\gamma\ d\sigma\ d\alpha\ dx \\
&\lesssim \|f\|_{\dot H^3}\int \frac{1}{\alpha^4}\|\delta_\alpha f_x\|_{L^\infty}\int_0^\alpha \frac{\|s_\eta f_x\|_{L^2}}{\eta^2}\eta^2\ d\eta\ d\alpha \\
&\lesssim \|f\|_{\dot H^3}\int \frac{\|\delta_\alpha f_x\|_{L^\infty}}{\alpha^4}\left(\int_0^\alpha \frac{\|s_\eta f_x\|_{L^2}^2}{\eta^4}\ d\eta\right)^{\frac{1}{2}}\left(\int_0^\alpha \eta^4\ d\eta\right)^{\frac{1}{2}}\ d\alpha \\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{\frac{5}{2}}}\int \frac{1}{|\alpha|^{\frac{3}{2}}}\|\delta_\alpha f_x\|_{L^\infty}\ d\alpha \\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{\frac{5}{2}}}\|f\|_{\dot B^{\frac{3}{2}}_{\infty,1}} \\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{\frac{5}{2}}}\|f\|^{\frac{1}{2}}_{\dot B^1_{\infty,\infty}}\|f\|^{\frac{1}{2}}_{\dot B^2_{\infty,\infty}}
\end{aligned}
$$

$$
\begin{aligned}
T_{4,1,1} &\lesssim \|f\|_{\dot{H}^3} \|f\|_{\dot{H}^{\frac{5}{2}}} \|f\|_{\dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \|f\|_{\dot{H}^{\frac{5}{2}}}^{\frac{1}{2}} \\
&\lesssim \|f\|_{\dot{H}^3} \|f\|_{\dot{H}^{\frac{5}{2}}}^{\frac{3}{2}} \|f\|_{\dot{H}^{\frac{3}{2}}}^{\frac{1}{2}}
\end{aligned}
$$

where we used $\dot{B}^{\frac{3}{2}}_{\infty,1} = \left[\dot{B}^{1}_{\infty,\infty}, \dot{B}^{2}_{\infty,\infty}\right]_{\frac{1}{2},\frac{1}{2}}$ and that $\left[\dot{H}^{3}, \dot{H}^{\frac{3}{2}}\right]_{\frac{2}{3},\frac{1}{3}} = \dot{H}^{\frac{5}{2}} \hookrightarrow \dot{B}^{2}_{\infty,\infty}$. Then, since

$$
\|f\|_{\dot{H}^{\frac{5}{2}}}^{\frac{3}{2}} \leq \|f\|_{\dot{H}^3} \|f\|_{\dot{H}^{\frac{3}{2}}}^{\frac{1}{2}}
$$

we conclude that

$$
T_{4,1,1} \lesssim \|f\|_{\dot{H}^3}^{2} \|f\|_{\dot{H}^{\frac{3}{2}}}
$$

Only one term seems **not** to be controlled as the others ($i.e. \lesssim P(\|f\|_{\dot{H}^{\frac{3}{2}}})\|f\|^2_{\dot{H}^3}$). The term is

$$
\begin{aligned}
T_{6,1,1,4} \quad = \quad & -\frac{1}{2\pi}\int\int\int_0^\infty\int_0^\infty \sigma e^{-\gamma-\sigma}\ \Lambda^3 f\ S_\alpha f \sin(\frac{\gamma}{2}D_\alpha f) \\
& \times \frac{\delta_\alpha f_{xx}}{\alpha}\ \tau_\alpha f_{xx} \sin(\sigma\tau_\alpha f_x)\cos(\arctan(\tau_\alpha f_x))\ d\alpha\ d\gamma\ d\sigma\ dx
\end{aligned}
$$

This term may be controlled as follows

$$
\begin{aligned}
T_{6,1,1,4} \quad & \lesssim \quad \|\Lambda^3 f\|_{L^2} \int \frac{\|s_\alpha f\|_{L^\infty}}{|\alpha|^2}\|\delta_\alpha f_{xx}\|_{L^4}\|\tau_\alpha f_{xx}\|_{L^4}\ d\alpha \\
& \lesssim \quad \|\Lambda^3 f\|_{L^2}\ \|f_{xx}\|^2_{L^4}\int \frac{\|s_\alpha f\|_{L^\infty}}{|\alpha|^2}\ d\alpha \\
& \lesssim \quad \|\Lambda^3 f\|_{L^2}\ \|f\|^2_{\dot{H}^{\frac{9}{4}}}\int \frac{\|s_\alpha f\|_{L^\infty}}{|\alpha|^2}\ d\alpha \\
& \lesssim \quad \|f\|^2_{\dot{H}^3}\ \|f\|_{\dot{H}^{\frac{3}{2}}}\|f\|_{\dot{B}^1_{\infty,1}}
\end{aligned}
$$

After many computations, we find

$$\frac{1}{2}\partial_t \|f\|^2_{\dot{H}^{3/2}} + \int \frac{|\Lambda^3 f|^2}{(1+|f_x|^2)^{\frac{3}{2}}}\, dx \lesssim \|f\|^2_{\dot{H}^3} \left(\mathcal{P}(\|f\|_{\dot{H}^{3/2}}) + \|f\|_{\dot{H}^{3/2}} \|f\|_{\dot{B}^1_{\infty,1}} \right)$$

Integrating in time gives

$$\|f(T)\|^2_{\dot{H}^{3/2}} + \int_0^T \int \frac{|\Lambda^3 f|^2}{(1+|f_x|^2)^{\frac{3}{2}}}\, dx\ ds \lesssim \|f_0\|^2_{\dot{H}^{3/2}}$$

$$+ \int_0^T \|f\|^2_{\dot{H}^3} \left(\mathcal{P}(\|f\|_{\dot{H}^{3/2}}) + \|f\|_{\dot{H}^{3/2}} \|f\|_{\dot{B}^1_{\infty,1}} \right)\, ds,$$

where $\mathcal{P}(X) = X + X^3$.

Then, since $0 \leq a \mapsto \frac{1}{(1+a^2)^{\frac{3}{2}}}$ is decreasing, we infer that

$$\frac{1}{(1+|f_x|^2)^{\frac{3}{2}}} \geq \frac{1}{(1+\|f_x\|_{L^\infty}^2)^{\frac{3}{2}}}$$

Finally,

$$\|f(T)\|_{\dot{H}^{3/2}}^2 + \int_0^T \frac{\|f\|_{\dot{H}^3}^2}{(1+L^2)^{\frac{3}{2}}}\, ds \lesssim \|f_0\|_{\dot{H}^{3/2}}^2$$
$$+ \int_0^T \|f\|_{\dot{H}^3}^2 \left(\mathcal{P}(\|f\|_{\dot{H}^{3/2}}) + \|f\|_{\dot{H}^{3/2}}\|f\|_{\dot{B}^1_{\infty,1}}\right)\, ds$$

Set $L := \|f_x\|_{L^\infty_x}$.

Thanks for your attention !

& Joyeux anniversaire Pierre Gilles !