INDEPENDENCE, from Kolmogorov to the construction of crystalline measures

Yves Meyer

Paris, November 8, 2024

- Independent random variables on the n-dimensional torus.
- Aleksandrov-Clark measures and Ahern measures.
- Pointwise products between n independent lighthouses yield positive crystalline measures on $\mathbb{R}^n$ (P.Kurasov and P.Sarnak).

- Independent random variables on the n-dimensional torus.
- Aleksandrov-Clark measures and Ahern measures.
- Pointwise products between n independent lighthouses yield positive crystalline measures on $\mathbb{R}^n$ (P.Kurasov and P.Sarnak).

- Independent random variables on the *n*-dimensional torus.
- Aleksandrov-Clark measures and Ahern measures.
- Pointwise products between *n* independent lighthouses yield positive crystalline measures on $\mathbb{R}^n$ (P.Kurasov and P.Sarnak).

Kolmogorov's definition of independence

In his famous book **Foundations of the theory of probability** Kolmogorov wrote:

The purpose of this monograph is to give an axiomatic foundation for the theory of probability. . . . This task would have been a rather hopeless one before the introduction of Lebesgue's theories of measure and integration.

However, after Lebesgue's publication of his investigations, the analogies between measure of a set and probability of an event, and between integral of a function and mathematical expectation of a random variable, became apparent.

These analogies allowed further extensions; **thus for example various properties of independent random variables were seen to be in complete analogy with the corresponding properties of orthogonal functions.**

Burkholder-Gundy

- In the sixties Paul-André Meyer and I were trying to bridge the gap between martingales and Littlewood-Paley decompositions.
- We failed but it led to the famous Burkholder-Gundy inequalities.

Riesz products

- A beautiful example is given by a Riesz product $\mu = \prod_0^\infty (1 + r \cos 3^k x),\ -1 \le r \le 1.$
- Lacunary implies independence.
- The integral of the product is the product of the integrals.
- Another example is given by *Le principe des soucoupes.*

Independent random variables on the *n*-dimensional torus

Let **T** denote the group of complex numbers z of modulus 1 and let $\mathbf{T^n}$ be the n-dimensional torus.

Then $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and **T** are identified by the canonical isomorphism $\theta \mapsto \exp 2\pi i\theta$.

The Fourier series expansion of a function $f \in L^2(\mathbb{T})$ is $f(x) = \sum_{k\in\mathbb{Z}} a(k) \exp 2\pi ikx$ while the Fourier series expansion of a function $f \in L^2(\mathbf{T})$ is $f(z) = \sum_{k\in\mathbb{Z}} a(k)z^k$.

Generalized inner functions

A proper cone $S \subset \mathbb{R}^n$ is a closed convex cone which does not contain a full line.

Definition

Let $S \subset \mathbb{R}^n$ be a proper cone. We denote by L^2_S the subspace of $L^2(\mathbb{T}^n)$ consisting of all square integrable functions on the n-dimensional torus whose Fourier coefficients vanish outside S. We write $f \in \mathcal{I}_S$ if $f \in L^2_S$ and if $|f(x)| = 1$ almost everywhere on $\mathbb{T}^n$. If S is a proper cone and if $f \in \mathcal{I}_S$ then f is called a generalized inner function.

A trivial example is $f(z) = z_1^{m_1} \cdots z_n^{m_n}, z_j = \exp(2\pi i\theta_j)$, when $m = (m_1, \cdots, m_n) \in S$.

Lemma

Let S be a proper cone. Then $\mathcal{I}_S$ is closed under pointwise multiplication. Moreover $f \in \mathcal{I}_S$ implies $f^{-1} = \bar{f} \in \mathcal{I}_{(-S)}$.

This is obvious since $S + S \subset S$.

If $H = \{x; x_j \geq 0, 1 \leq j \leq n\}$ then $\mathcal{I}_H$ is the set of standard inner function on $\mathbb{T}^n$. Let $\mathbb{D}^n \subset \mathbb{C}^n$ be the polydisc defined by $|z_j| \leq 1, 1 \leq j \leq n$. Then $\mathbf{T^n} \subset \mathbb{C}^\mathbf{n}$ is the distinguished boundary of $\mathbb{D}^n$.

Lemma

We have $f \in \mathcal{I}_H$ if and only if $f(z)$ is the trace on $\mathbf{T^n}$ of an holomorphic function $F \in H^\infty(\mathbb{D}^n)$ and if $|f(z)| = 1$ almost everywhere on $\mathbf{T^n}$.

Strictly speaking $f \in \mathcal{I}_H$ is defined on $\mathbb{T}^n$ and shall be moved on $\mathbf{T^n}$.

- Walter Rudin and E.L. Stout proved that any smooth standard inner function f on $\mathbf{T^n}$ is a rational function: $f = Q/P$ where P and Q are two polynomials and P does not vanish on $\mathbb{D}^n$.
- Moreover we have $Q(z) = M(z)P^*(1/z)$ where M is a monomial and the coefficients of the polynomial P^* are conjugates of the coefficients of P.
- Finally $1/z = (1/z_1, ..., 1/z_m)$.
- It implies that any smooth generalized inner function is a rational function.

Independent proper cones

Definition

Let $2 \leq m \leq n$. The proper cones $S_1, \ldots, S_m$ are independent over $\mathbb{Q}$ if the following property holds: whenever m vectors $x(1) \in \mathbb{Z}^n, \ldots, x(m) \in \mathbb{Z}^n$, satisfy

$$x(j) \in S_j, \ x(j) \neq 0, \ 1 \leq j \leq m, \tag{1}$$

then these vectors $x(1), \ldots, x(m)$ are linearly independent over $\mathbb{Q}$.

If $S_1, \ldots, S_m$ are m independent proper cones, then $\pm S_1, \ldots, \pm S_m$ are also m independent proper cones.

A stronger definition of independence is the following:

Definition

Let $2 \leq m \leq n$. The proper cones $S_1, \ldots, S_m$ are independent over $\mathbb{R}$ if the following property holds: whenever m vectors $x(1) \in \mathbb{R}^n, \ldots, x(m) \in \mathbb{R}^n$, satisfy $x(j) \in S_j,\ x(j) \neq 0,\ 1 \leq j \leq m$, then these vectors $x(1), \ldots, x(m)$ are linearly independent over $\mathbb{R}$.

The theorem

The n-dimensional torus is now viewed as a probability space where the probability measure is the Haar measure dx normalized by $\int_{\mathbb{T}^n} d\,x = 1$.

Theorem

Let $2 \leq m \leq n$ and let $S_1, \dots, S_m$ be m proper cones. Then the following two properties are equivalent ones:

(a) *The m proper cones $S_1, \dots, S_m$ are independent over $\mathbb{Q}$.*

(b) *$\forall f_1 \in \mathcal{I}_{S_1}, \dots, \forall f_m \in \mathcal{I}_{S_m}$ these inner functions $f_1, \dots, f_m$ are m independent random variables on $\mathbb{T}^n$.*

Loewner's lemma

If $\int_{\mathbb{T}^n} f_j(x)\,dx = 0, 1 \leq j \leq m$, we have a stronger result. The random variables $f_1, \ldots, f_m$ are independent and identically distributed on $\mathbb{T}^n$ and the pushforward measure of the Haar measure on $\mathbb{T}^n$ by $F = (f_1, \ldots, f_m)$ is the Haar measure on $\mathbf{T}^m$. Loewner's lemma (Lemma 3) is the one dimensional case $(m = 1)$ of this result.

Lemma

Let f be a standard inner function on $\mathbb{T}^n$ such that $f(0) = 0$ and let g be a continuous function on $\mathbf{T}$. Let $d\lambda_n$ be the Haar measure on $\mathbb{T}^n$. Then

$$\int_{\mathbb{T}^n} g \circ f \, d\lambda_n = \int_{\mathbf{T}} g \, d\lambda_1. \tag{2}$$

The functions $f(z) = \frac{3z_1 z_2^2 + z_1 + z_2^2}{3 + z_1 + z_2^2}$ and $g(z) = z_1/z_2$ are two independent random variables on $\mathbf{T}^2$.

Indeed in this example the cone S is the first quadrant while T is defined by $x_2 = -x_1$ and $x_1 \geq 0$.

Construction of Ahern measures on $\mathbb{T}^n$.

A proper double cone $T \subset \mathbb{R}^n$ is defined as $T = S \cup (-S)$ where S is a proper cone.

Definition

A signed measure μ on $\mathbb{T}^n$ is an Ahern measure if there exists a proper double cone T such that for any $k \in \mathbb{Z}^n$ we have $\widehat{\mu}(k) = 0$ unless $k \in T$.

Aleksandrov-Clark measures are Ahern measures.

Aleksandrov-Clark measures are Ahern measures

Let S be a proper cone and let $J \in \mathcal{I}_S$ be a generalized inner function on $\mathbb{T}^n$. Let $\gamma = \int_{\mathbb{T}^n} J(x)\,dx$. We obviously have $|\gamma| \leq 1$. If $|\gamma| = 1$ then J is a constant function. From now on this is excluded.

Theorem

Let $J \in \mathcal{I}_S$ be a generalized inner function on $\mathbb{T}^n$. Let $\gamma = \int_{\mathbb{T}^n} J(x)\,dx$ and let us assume that $|\gamma| < 1$. Let ν be a Radon measure on $\mathbf{T}$, Then $\mu = \nu \circ J$ is a Ahern measure on $\mathbb{T}^n$ and

$$\|\nu \circ J\| \leq \frac{1+|\gamma|}{1-|\gamma|}\|\nu\|. \tag{6}$$

- Here $\|\nu\|$ denotes the total mass of the measure ν.
- When J is a standard inner function on the unit circle $\mathbb{T}$ and when ν is the Dirac measure at $\tau, |\tau| = 1$, then $\mu_\tau = \nu \circ J$ is a standard Aleksandov-Clark measure [2].
- We first define $\nu \circ J$.

Definition

Let $\sum_{k\in\mathbb{Z}} c_k z^k$ be the Fourier series expansion of ν. Then the series

$$\sum_{k\in\mathbb{Z}} c_k J^k \tag{7}$$

converges in the distributional sense to a Radon measure on $\mathbb{T}^n$ denoted by $\nu \circ J$.

- We now show that μ is an Ahern measure.

Lemma

Let $J \in \mathcal{I}_S$ be a generalized inner function on $\mathbb{T}^n$ and let ν be a positive Radon measure on **T**. *Then the Fourier coefficients $\widehat{\mu}(m)$ of $\mu = \nu \circ J$ vanish if $m \notin S \cup (-S)$.*

Indeed we have $\mu = \nu \circ J = \sum_{k \in \mathbb{Z}} c_k J^k$. But the Fourier coefficients of J^k vanish outside S if $k \geq 1$ and outside $-S$ if $k \leq -1$ which ends the proof.

Geometrical structure of Aleksandrov-Clark measures

Lemma

Let J be a smooth inner function and let K be the compact support of ν. Then the measure $\mu = \nu \circ J$ is supported by $U_K = \{x; J(x) \in K\}$.

Lemma

Let $J \in \mathcal{C}^\infty(\mathbb{T}^n, \mathbf{T})$. Let $|\tau| = 1$ and let us assume that $\nabla J(x) \neq 0$ everywhere on the level set $U_\tau = \{x : J(x) = \tau\}$. Then the measure $\mu_\tau = \delta_\tau \circ J$ makes sense, it is absolutely continuous with respect to the surface measure $d\sigma$ on U_τ and we have $\mu_\tau = |\nabla J|^{-1} d\sigma$.

Construction of crystalline measures

Definition

A “lighthouse" [6] is a **positive** Radon measure on $\mathbb{R}^n$ which is supported by a closed set F of zero Lebesgue measure and whose distributional Fourier transform is supported by a proper double cone T.

- A proper double cone T is defined as $T = S \cup (-S)$ where S is a proper cone.
- We now consider the Aleksander-Clark measure $\mu_\tau = \delta_\tau \circ J$ of the preceding Lemma as a periodic measure on $\mathbb{R}^n$. If the measure of the level set U_τ is 0 then μ_τ is a lighthouse.

Turning around

Lemma

Let $T \subset \mathbb{R}^n$ be an arbitrary proper cone. Then for any $A \in GL(n, \mathbb{R})$ and any lighthouse (μ, T) the measure $\mu \circ A$ is a lighthouse and its Fourier transform is supported by $A^*(T)$.

Pointwise product between m weakly independent lighthouses

Definition

Let $2 \leq m \leq n$. The lighthouses $(\mu_1, T_1), \cdots, (\mu_m, T_m)$ are weakly independent if the proper double cones $T_1, \cdots, T_m$ are independent over $\mathbb{R}$.

Theorem

Let $2 \leq m \leq n$. Let $(\mu_1, T_1), \ldots, (\mu_m, T_m)$ be m weakly independent lighthouses. Then the pointwise product $\mu = \mu_1 \cdots \mu_m$ is a positive measure. The support of μ is contained in the intersection $\cap_1^m F_j$ where F_j is the closed support of μ_j. Moreover if each $\widehat{\mu}_j, 1 \leq j \leq m$, is an atomic measure supported by a locally finite set then the same is true for $\widehat{\mu}$.

Crystalline measures

A crystalline measure is an atomic measure μ enjoying the following three properties: (*a*) the support of μ is locally finite, (*b*) μ is a tempered distribution and (*c*) the distributional Fourier transform of μ is also an atomic measure carried by a locally finite set.

Recipe

- We start with a standard smooth inner function J.
- We consider the corresponding Aleksandrov-Clark measure $\mu = \delta_\tau \circ J$.
- We use the turning Lemma to obtain n independent proper cones $A_j^*(T)$.
- We consider $\mu_j = \mu \circ A_j$.

Theorem

The pointwise product $\mu = \mu_1 \cdots \mu_n$ is a crystalline measure if and only if the support of μ is locally finite.

This geometric condition is easily checked since these μ_j are Aleksandrov-Clark measures.

Hörmander's seminal work

The pointwise product $u\,v$ between two tempered distributions $u, v \in \mathcal{S}'(\mathbb{R}^n)$ does not exist in general. However this product exists if the two distributions u and v are independent in a sense given by two-microlocal analysis. Here is a tentative definition of the pointwise product of two tempered distributions. Let φ and ψ be two functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that $\int \varphi = 1$, $\int \psi = 1$, and let φ_ϵ, $0 < \epsilon \leq 1$, be given by $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. The family ψ_ϵ, $0 < \epsilon \leq 1$, is defined similarly.

Hörmander's seminal work

Definition

Let u and v be two tempered distributions. Let us assume that (i) the pointwise product $(u * \varphi_\epsilon)(v * \psi_\epsilon)$ tends to a limit in $\mathcal{S}'(\mathbb{R}^n)$ as ϵ tends to 0 and (ii) that this limit does not depend on the choices of φ and ψ. Then the pointwise product uv between u and v exists and is defined by

$$u\,v = \lim_{\epsilon \to 0} (u * \varphi_\epsilon)(v * \psi_\epsilon). \tag{8}$$

Hörmander's seminal work

Theorem

Let S and T be two closed cones such that $S \cap (-T) = \{0\}$. If u and v are two tempered distributions on $\mathbb{R}^n$, if $\widehat{u}$ is supported by S, and if $\widehat{v}$ is supported by T then the pointwise product uv makes sense. Moreover if $u_j \rightharpoonup u$ and $v_j \rightharpoonup v$ then $u_j v_j \rightharpoonup uv$.

- P.Ahern. *Inner functions in the polydisc and measures on the torus.* Michigan Math. J. 20 (1973).
- D.Clark. *One dimensional perturbations of restricted shifts.* J. Anal. Math. 25 (1972) 169–191.
- L.Hörmander. The analysis of linear partial differential operators: I Distribution Theory and Fourier Analysis 2nd edn (1990) Springer.
- A.Kolmogorov. *Foundation of the Theory of Probability.*

- P.Kurasov and P.Sarnak. *Stable polynomials and crystalline measures.* J. Math. Phys. 61, 083501 (2020)
- Y.Meyer *Crystalline measures and wave front sets.* Volume in hommage to Guido Weiss.
- Y.Meyer *Multidimensional crystalline measures.* Trans. R. Norw. Soc. Sci. Lette. (2023) 1-24.
- W.Rudin. *Function theory in the polydiscs*. Benjamin, New York, 1969.
- W.Rudin and L.E. Stout. *Boundary properties of functions of several complex variables.* Journal of Mathematics and Mechanics. (1965) 991-1005.