Boundary effects in the Euler equations and in the 0 viscosity limit of solutions of Navier Stokes equations with no slip boundary condition. In collaboration with E. Titi , D.Boutros , T. Nguyen and TR. Nguyen Conférence dédiée à Pierre-Gilles Lemarié-Rieusset.

Claude Bardos, Retired . https://www.ljll.math.upmc.fr/ ˜ bardos:

November 4, 2024

Some relations existing in the mathematical approaches in fluid dynamic of the following issues.

1. Energy conservation / Anomalous energy dissipation,
2. Onsager Conjecture,
3. Two critical numbers $0 < \frac{1}{3} \leq \frac{1}{2}$,
4. Boundary effect, Kato criteria , Prandlt Von Karman turbulent boundary layer,
5. Analytic Prandlt Boundary layer and Gortler vortices.

In $\Omega \times \mathbb{R}_t \quad \partial_t u + u \cdot \nabla u + \nabla p = 0\,, \qquad \nabla \cdot u = 0\,.$

Or for weak solutions $\quad \partial_t u + \nabla : (u \otimes u) + \nabla p = 0\,;$

With boundary on $\quad \partial\Omega \times \mathbb{R}_t \qquad u \cdot \vec{n} = 0\,;$

And "*formally*" energy conservation $\displaystyle \frac{d}{dt}\int_\Omega |u(x,t)|^2 dx = 0\,.$

Formally refer to the Onsager conjecture (1949) which relates the anomalous energy dissipation with a loss in regularity up to the order $\frac{1}{3}$ (in Holder spaces or avatars..)

In the absence of boundary:

- Solutions in $L^3(0, T; C^{0,\alpha})$ with $\alpha > \frac{1}{3}$ conserve the energy (Constantin, E, Titi 1994.).

$$-\Delta p = (\nabla \otimes \nabla) : (u \otimes u) \quad \Rightarrow p \in C^{0,\alpha}\,. \qquad (1)$$

- There exist solutions belonging to $L^3(0, T; C^{0,\alpha})$ with $\alpha < \frac{1}{3}$ that dissipate the energy: C.DeLellis and L.SzekelyhidiJr. 2009. Then Isett 2018.

With $\vec{n}$ an extension of the "ingoing" normal near $\partial\Omega$:

$$\text{In} \quad \Omega \quad -\Delta p = (\nabla \otimes \nabla) : (u \otimes u) \tag{2}$$

For the boundary:

$$\begin{aligned} &\text{In } \Omega\,,\ \partial_t(u\cdot\vec{n}) + [\nabla\cdot(u\otimes u)]\cdot n + \nabla p\cdot\vec{n} = 0 \\ &\Rightarrow \text{On } \partial\Omega\,, \quad [\nabla\cdot(u\otimes u)]\cdot\vec{n} + \nabla p\cdot\vec{n} = 0\,. \end{aligned} \tag{3}$$

$$\begin{aligned} &\text{Formally on } \partial\Omega \quad u\cdot\vec{n} = 0 \Rightarrow \partial_{\vec{n}}(u\cdot\vec{n})^2 = 0 \\ &\Rightarrow \partial_{\vec{n}} p = -[\nabla\cdot(u\otimes u)]\cdot[\nabla\cdot(u\otimes u)]\cdot\vec{n} = (u\otimes u) : \nabla\vec{n}\,. \end{aligned} \tag{4}$$

The standard Holder elliptic regularity gives for $k > 1$ and *formally* for $k = 0$.

$$u \in C^{k,\alpha}(\Omega) \Rightarrow p \in C^{k,\alpha}(\Omega)\,. \tag{5}$$

- From the $C^{0,\alpha}$ regularity of the field u follows the *interior* $C^{0,2\alpha}$ regularity of the of the pressure (cf. Below).

With an extra hypothesis near a small neighbourhood of the boundary on the pressure the conservation of energy follows CB, Titi 2018.

- As shown by an example below this extra hypothesis on the pressure near the boundary does not follow from the basic equations (2) (4).

The reason being that under the sole regularity $u \in C^{0,\alpha}(\Omega)$ with $\alpha < \frac{1}{2}$ the expression $\partial_{\vec{n}}(u \cdot \vec{n})^2$ makes no sense on the boundary.

- However with the introduction of a modified *very weak form of the boundary condition* appearing in (4) one shows that the pressure is in $C^{0,\alpha}(\Omega)$ up to the boundary.
- With such considerations one obtains the energy conservation under the unique $C^{0,\alpha}(\Omega)$ regularity of the field u.

In $\Omega=\mathbb{R}^d$ or $\mathbb{T}^d$ a pressure p defined in term of a divergence free vector field u by the formula:

$$-\Delta u=(\nabla\otimes\nabla)(u\otimes u)$$

inherit from u the following properties:

$$\begin{aligned}
&\Omega=\mathbb{R}^d \text{ or } \mathbb{T}^d\,,\quad u\in C^{0,\alpha}(\Omega)\cap L^1(\Omega)\quad \nabla\cdot u=0\\
&\qquad\qquad\Rightarrow\\
&\text{For } 0<\alpha<1\quad \alpha\neq\frac{1}{2}\quad \Rightarrow \|p\|_{C^{0,2\alpha}}\leq C\|u\|^2_{C^{0,\alpha}}\,.\\
&\text{For } \alpha>1\,, C^{2\alpha}=C^{1,\alpha-1}\,.\\
&\text{For } \alpha=\frac{1}{2}\quad p\in \mathrm{Lip}_{\mathrm{Log}}=\{f\quad \frac{|f(x)-f(y)|}{|x-y|\log(\frac{1}{|x-y|})}<\infty\}\,.
\end{aligned}\tag{6}$$

LipLog or Lizorkin spaces.

(2010) Proof Sylvestre in $\mathbb{R}^d$. Also L. De Rosa, M. Latocca, and G. Stefani. 2023.

$$
\begin{aligned}
&-\Delta G = \delta \text{ Green function of } -\Delta \\
&d = 2 \Rightarrow G(y) = \frac{1}{2\pi} \log \frac{1}{|y|} \quad d > 2 \Rightarrow G(y) = \frac{c_d}{|y|^{d-2}} , \\
&\phi(y) = G(y - x_1) - G(y - x_2) \, , \partial_{ij} G(y) = \frac{|y|^2 \delta_{ij} - 2 y_i y_j}{|y|^{d+2}} , \\
&p(x_1) - p(x_2) = \int p(y) \Delta \phi(y) dy \\
&= \int (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) \partial_i \partial_j \phi(y) dy \, .
\end{aligned}
$$

The above term is “quadratic” in α” .

Introduce: $\overline{x} = \frac{x_1+x_2}{2} \quad r = |x_1 + x_2|$

$$|\int (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2))\partial_i\partial_j\phi(y)dy| \leq \int_{|y-\overline{x}|>5r} \ldots + \int_{|y-\overline{x}|<5r} \ldots.$$

For the first term use:

$$|y - \overline{x}| > 5r \Rightarrow |\partial_i\partial_j\phi(y)\| \leq \frac{Cr}{|y - \overline{x}|^{d+1}} \Rightarrow$$

$$\int_{|y-\overline{x}|>5r} \ldots\ldots \leq C(\|u\|^{\alpha})^2 \int_{|y-\overline{x}|>5r} |y - x_1|^{\alpha}|y - x_2|^{\alpha} \frac{Cr}{|y - \overline{x}|^{d+1}} dy$$

The second is bounded by the sum of two terms of the same order (for $k = 1, 2$)

$$\int_{|y-\overline{x}|<5r} (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2))|\partial_i\partial_j G(y - x_k)|dy$$

$$\leq C(\|u\|^{\alpha}) \int_{|y-\overline{x}|<5r} |y - x_k|^{\alpha} \frac{1}{|y - x_k|^{d}} dy\,.$$

And the estimate follows.

Shows that the standard regularity, Onsager type $C^{0,\alpha}$, $\frac{1}{3} < \alpha < \frac{1}{2}$ hypothesis on the fluid are not by themselves sufficient to define $\partial_{\vec{n}} p$ on $\partial\Omega$.

$$\text{In} \quad \mathbb{R}^+_x \times \mathbb{T}_y \quad U(x,y) = (u(x,y), v(x,y)) \quad \text{and} \quad y \mapsto \theta(y) \in \mathcal{D}(\mathbb{T})$$

$$u(x,y) = -\lim_{N\to\infty} \sum_{0\le k\le N} 2^{-\alpha k} \sin(2^k \pi x) \cos(2^k \pi y)\,,$$

$$v(x,y) = \lim_{N\to\infty} \sum_{0\le k\le N} 2^{-\alpha k} \cos(2^k \pi x) \sin(2^k \pi y)\,,$$

$$\frac{1}{x} \int u(x,y)^2 \theta(y) dy = \frac{1}{x} \lim_{N\to\infty} \sum_{0\le k\le N, 0\le l\le N} 2^{-\alpha(k+l)} \sin(2^k \pi x) \sin(2^l \pi x)$$
$$\int \cos(2^k \pi y) \cos(2^l \pi y) \theta(y) dy\,.$$

Modulo the non resonant termes:

$$\frac{1}{x}\int u(x,y)^2\theta(y)dy =$$

$$\frac{1}{2x}\lim_{N\to\infty}\sum_{0\le k\le N} 2^{-2\alpha k}(\sin(2^k\pi x))^2\int 1+(\cos(2^{k+1}\pi y))^2\theta(y)dy+\mathcal{R}_1$$

$$=\frac{1}{x}\lim_{N\to\infty}\sum_{0\le k\le N} 2^{-2\alpha k}(\sin(2^k\pi x))^2\int \theta(y)dy+\mathcal{R}_2$$

As a consequence inserting the value $x_K=\frac{1}{2^{K+1}}$ in the above sum one has:

$$\frac{1}{x_K}|\int (u(x_K,y))^2\theta(y)dy| \ge 2^{K(1-2\alpha)}|2\int \theta(y)dy|\,.$$

The normal derivative of $(u\cdot\vec{n})^2$ hence of p is not defined on $\partial\Omega$ and also on any sub manfold of Ω.

Ω is an open set with a smooth (C^4 boundary $\partial\Omega$.)

$$d(x,\partial\Omega)<\delta \Rightarrow d(x,\partial\Omega)=\|x-\overline{x}\|\,,\overline{x}\in\partial\Omega$$
$$x\mapsto\overline{x}\in C^2(\Omega)\,,x\mapsto d(x,\partial\Omega)\in C^4(\Omega)\,.$$

Introduces *a boundary layer localisation*:

$$x\mapsto\phi(x)\in\mathcal{C}^2(\mathbb{R}^d)\,,$$
$$d(x,\partial\Omega)>\delta\Rightarrow\phi(x)=0\,,$$
$$d(x,\partial\Omega)<\frac{\delta}{2}\Rightarrow\phi(x)=1\,,$$
$$-\Delta\big(p+\phi(x)(u\cdot n)^2\big)=(\nabla\otimes\nabla):(u\otimes u)-\Delta(\phi(x)(u\cdot n)^2)\quad\text{in }\Omega,$$
$$\partial_n\big(p+\phi(x)(u\cdot n)^2\big)=\big(u\otimes u\big):\nabla n$$

in a very weak sense: in $H^{-2}(\partial\Omega\times(0,T)))$.

$$-\Delta P = (\nabla \otimes \nabla) : (u \otimes u) - \Delta(\phi(x)(u \cdot n)^2), \tag{7}$$

which is satisfied in the sense of distributions. In particular, it means that for test functions $\psi \in \mathcal{D}(\Omega)$

$$-\int_\Omega P \Delta \psi dx = \int_\Omega u_i u_j \partial_i \partial_j \psi dx - \int_\Omega \phi(x)(u \cdot n)^2 \Delta \psi dx. \tag{8}$$

$$-\Delta P = (\nabla \otimes \nabla) : (u \otimes u) - \Delta(\phi(x)(u \cdot n)^2), \tag{9}$$

which is satisfied in the sense of distributions. In particular, it means that for test functions $\psi \in \mathcal{D}(\Omega)$

$$-\int_\Omega P \Delta \psi dx = \int_\Omega u_i u_j \partial_i \partial_j \psi dx - \int_\Omega \phi(x)(u \cdot n)^2 \Delta \psi dx. \tag{10}$$

$$\text{In } H^{-2}(\partial\Omega \times (0,T))), \ \partial_n P = \big(u \otimes u\big) : \nabla n \quad \text{on } \partial\Omega, \tag{11}$$

As a consequence P belongs to $C^{0,\alpha}$ and since $(u \otimes u)$ belongs to the same space one has also for $\alpha > \frac{1}{3}$ $\quad p \in C^{0,\alpha}$. C.B. D. Boutros, E. Titi, arXiv:2304.01952. 2023.

- Adapted to the description of boundary effect, including incoming flux.
- The simplest but the most rigide boundary is the no slip boundary condition $u_\nu = 0$
- For small (realistic $\nu \to 0+$) and convergence to solution of the Euler equation the tangential component of the velocity does remain equal to 0 A boundary layer appear and Prandlt 1905 (the main model of further boundary layer analysis over the past century).
- This boundary layer is related to generation of "turbulence / anomalous energy dissipation."
- However Prandlt-Von Karman 1930 observed that the turbulence is generated in a domain of size $\nu << \sqrt{\nu}$.
- Below comparison at the level of initial valuee problems with smooth solutions of the Euler equation, "lipschitz" for the absence of anomalous energy dissipation and analytic for sharper analysis.

E. Titi and C.B. An avatar Kato Thoerem , based on simple Gronwall estimate .

Theorem

(In dimension 2 and 3) Let u be weak solution to the Euler equations in $[0, T] \times \Omega$ *satisfying* $\|\nabla u\|_{L^\infty([0,T]\times\Omega)} < \infty$. *Consider* $(\nu > 0\,, u_\nu)$ *Leray weak solutions to the Navier-Stokes :*

$$\frac{1}{2}\|u_\nu(t)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla_x u_\nu(t)\|^2_{L^2(\Omega)} dt \leq \frac{1}{2}\|u_\nu(0)\|^2_{L^2(\Omega)} \tag{12}$$

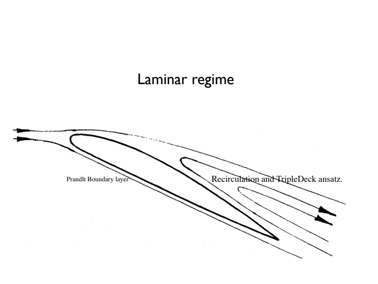
uniformly in $\nu \to 0$. *Assume that their vorticity* $\omega_\nu = \nabla^\perp \cdot u_\nu$ *satisfies*

$$\limsup_{\nu\to 0} \Big(- \int_0^T \int_{\partial\Omega} \nu\omega_\nu(t,\sigma) u(t,\sigma)\cdot\tau(\sigma) d\sigma dt \Big) = 0, \tag{13}$$

then any $\overline{u}_\nu$*, which is a* **weak**$-*$ **limit** *in* $L^\infty([0,T]; L^2(\Omega))$ *of a subsequence* u_{ν_j} *as* $\nu_j \to 0$*, satisfies the stability estimate (and convergence for* $\overline{u_\nu(0)} - u(0) = 0$).

For the Kato Theorem it is enough to have

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} \left((\frac{\partial u_\nu}{\partial \vec{n}}(\sigma, t))_\tau u(\sigma, t) \right) d\sigma dt = 0$$



Equivalents of Kato theorem:

$$\forall w(x,t) \in L^\infty((0,T)\times\partial\Omega) \quad \text{with} \quad w\cdot\vec{n}=0\,,$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}(\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau w(\sigma,t)d\sigma dt=0 \tag{16}$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}((\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau u_\tau(\sigma,t))_- d\sigma dt=0 \tag{17}$$

$$u_\nu(t)\to u(t) \text{ in } L^2(\Omega) \text{ uniformly in } t\in[0,T]\,, \tag{18}$$

$$u_\nu(t)\to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t\in[0,T]\,, \tag{19}$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_\Omega|\nabla u_\nu(x,t)|^2dxdt=0\,, \tag{20}$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\Omega\cap\{0<d(x,\partial\Omega)<\frac{\nu}{2}\}}|\nabla\wedge u_\nu(x,t)|^2dxdt=0\,. \tag{21}$$

$$\lim_{\nu\to 0}\frac{1}{\nu}\int_0^T\int_{\Omega\cap\{\frac{\nu}{4}<d(x,\partial\Omega)<\frac{\nu}{2}\}}|u_\nu(x,t)|^2dxdt=0\,. \tag{22}$$

Next our purpose is the most possibly direct proof of (13) for short time assuming that boundary of the domain and initial value of the solution are analytic and confirming the role of curvature in the loss of stability.

Theorem

Let $u_0(x)$ be an initial data that is analytic up to the boundary $\partial\Omega$ and vanishes on the boundary. Then, there is a positive time T, independent of ν, so that the unique solutions $u_\nu(t)$ to the Navier-Stokes problem satisfies the estimate

$$\lim_{\nu\to 0}\sqrt{\nu}\|\omega_\nu\|_{L^\infty([0,T]\times\partial\Omega))} < \infty. \tag{23}$$

Proof based on the extension to any domain with analytic curved boundary the following recent tools .

1. C. R. Anderson Vorticity boundary conditions and boundary vorticity generation for two-dimensional viscous incompressible flows. J. Comput. Phys. 1989.
2. Y. Maekawa, On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. Comm. Pure Appl. Math. (2014).
3. T.T. Nguyen and T.T. Nguyen. The inviscid limit of Navier-Stokes equations for analytic data on the half-space. Arch. Ration.Mech. Anal., 2018.
4. The release of the analyticity hypothesis away from the boundary I. Kukavica, V Vicol and F. Wang, Arch. Ration. Mech. Anal. (2020).
5. A well adapted localisation (geodesic) for the $C^{0\alpha}$ regularity of the pressure near the boundary (B. and Titi 2022, extended to $3d$ with D. Boutros)

On $\partial\Omega$ with $\partial_n = \tau \cdot \nabla^\perp$ one has , with $(-\Delta)^{-1}$ inverse of the Dirichlet Laplacian:

$$0 = \tau \cdot \partial_t u = \tau \cdot \nabla^\perp \Delta^{-1} \partial_t \omega = \partial_n [\Delta^{-1}(\nu \Delta \omega - u \cdot \nabla \omega)] \tag{24}$$

With Dirichlet Neumann operator and $\vec{n}$ interior normal

$$\begin{aligned} &\omega^* = \omega \quad \text{on } \partial\Omega\,, -\Delta\omega^* = 0, \text{in} \quad \Omega \quad DN(\omega) = -\partial_n \omega^*, \qquad \text{on} \quad \partial\Omega, \\ &\partial_n[\Delta^{-1}\Delta\omega] = \partial_n[\Delta^{-1}\Delta(\omega - \omega^*)] = (\partial_n + DN)\omega\,. \end{aligned} \tag{25}$$

$$\begin{aligned} &(\partial_n \omega_\nu + DN\omega_\nu) = \frac{1}{\nu} \partial_{\vec{n}} \Delta^{-1} (u \cdot \nabla \omega_\nu)\,, \\ &\partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0\,. \end{aligned} \tag{26}$$

Remark

The standard energy estimates :

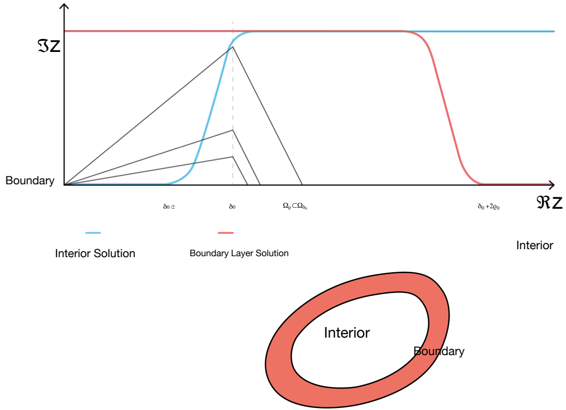
$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega |\omega_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla_\nu \omega_\nu(x,t)|^2 dx \\ = \int_{\partial\Omega} \nu DN\omega_\nu\omega_\nu d\sigma + \partial_{\vec{n}}\Delta^{-1}(u_\nu\cdot\nabla\omega_\nu)d\sigma\end{aligned} \tag{27}$$

indicates that the problem is ill posed even for $\nu > 0$ in any Sobolev space. However it is well posed in space of analytic functions. And ω_ν is analytic in $(t > 0, X + iY, X \in \Omega \times Y \in \mathbb{R}^2)$ while the solution of the Euler equation with analytic initial data is also analytic for

$$t \geq 0, X + iY, X \in \Omega \times |Y| \leq Ce^{-Ce^{Ct}}.$$

This is why such formulation may be well adapted (Anderson) for numerical computations (smooth data, small Reynolds number).

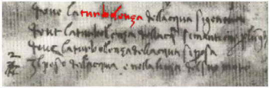
For proof decompose ω_ν in $\omega^b_\nu = \phi^b \omega_\nu$ defined in $(\theta, z) \in \mathbb{R}/(\mathbb{Z}\tilde{L}) \times \mathbb{R}^+_z)$ and $\omega^i_\nu = \phi^i \omega_\nu$ of compact support in $\mathbb{R}^2$



Changing t into $\lambda^2 t$ one has for rescaled variables $(\theta, z) \in \mathbb{R}/(\mathbb{Z}\tilde{L}) \times \mathbb{R}_z^+$:

$$\begin{aligned} &\partial_t \omega_\nu^b - \nu\Delta\omega_\nu^b = -\nu\lambda^2(m(z,\theta)\partial_\theta^2\omega^b) - \lambda^{-2}u_\nu^b \cdot \nabla\omega_\nu^b + \overline{K_1(\lambda)} \\ &\nu(\partial_n\omega_\nu^b + |\partial_\theta|\omega_\nu^b) = \lambda^{-1}[\partial_z\Delta^{-1}(u^b \cdot \nabla\omega_\nu^b)] - \nu B(\omega_\nu) + \overline{K_2(\lambda)}\,. \end{aligned} \tag{28}$$

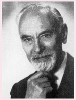
The role of λ is to "flatten " the curvature near the boundary where an explicit half space form of the *Stokes kernel is used.* In the change of variables $\theta \mapsto \lambda\theta$ the curvature is changed into $\lambda^3\gamma(\lambda\theta)$ this makes appear the coefficient λ^2 in front of $\lambda^2 m(z,\theta)\partial_\theta^2\omega^b$ which then can be dominated by the laplacian. This goes very well with the observation of vortices generated in the fluid by curved boundary "Gortler Vortices".

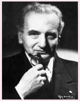


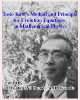
Where the 'turbolenza' of the water is generated.

Where the 'turbolenza' of the water persists for a long distance.

2. Where the 'turbolenza' of the water settles.







Pierre Gilles Many thanks for friendship contributions and patience.
Best wishes for continuation.