Vanishing viscosity limit for hyperbolic system in 1-d with nonlinear viscosity

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This talk is concerned with the vanishing viscosity limit for hyperbolic system of conservation laws. We consider the following parabolic approximation of the hyperbolic system

$$u_t + A(u)u_x = \varepsilon(B(u)u_x)_x \qquad \text{for } t > 0, x \in \mathbb{R}, \tag{1}$$

$$u(0,x) = \bar{u}(x) \qquad \text{for } x \in \mathbb{R}, \tag{2}$$

where $u : [0,+\infty) \times \mathbb{R} \to \mathbb{R}^n$ and A, B are $n \times n$ matrices satisfying the following conditions for some $\mathcal{U} \subset \mathbb{R}^n$.

1. Strict Hyperbolicity: The matrix $A(u)$ is C^3 function and has n distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$ for $u \in \mathcal{U}$.
2. The matrix $B(u)$ is a C^2 function and positive symmetric definite with $B(u) \geq c_0 \mathbb{I}_n$ for $u \in \mathcal{U}$ and for some $c_0 > 0$.
3. $$A(u)B(u) = B(u)A(u) \text{ for all } u \in \mathcal{U}. \tag{3}$$

   In particular $B(u)$ has n eigenvalues $(\mu_i(u))_{1 \leq i \leq n}$.

Remark

In the sequel we consider $l_1, \cdots, l_n$, $r_1, \cdots, r_n$ as left and right eigenvectors of $A(u)$ such that

$$\|r_i(u)\| = 1 \text{ and } l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4}$$

Remark

If we assume that $A(u) = Df(u)$ with f a regular function from $\mathbb{R}^n$ in $\mathbb{R}^n$, it is well-known [Glimm, Bressan et al] that the limit vanishing viscosity system which is conservative:

$$\begin{cases} u_t + \partial_x f(u) = 0, \\ u(0,\cdot) = \bar{u}, \end{cases}$$

has a unique global weak solution provided that u_0 is small in $TV(\mathbb{R})$. The class of uniqueness select some shock which satisfy the Lax condition.
Generally it is assumed that the fields are linearly degenerate ($\nabla\lambda_i(u)\cdot r_i(u) = 0$) or genuinely non linear ($\nabla\lambda_i(u)\cdot r_i(u) \neq 0$).
We can note that we are not able to prove the existence of global weak solution when the system is not conservative.

Some questions on the vanishing viscosity limit?

- Q1 Can we obtain the existence of global strong solution for the parabolic system which are uniformly bounded in $TV(\mathbb{R})$?
- Q2 Can we prove that the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges strongly to u ? When the system is conservative we can expect that u is a global weak solution of the limit vanishing viscosity system.
- Q3 Is it true that the limit u depends on the viscosity coefficients $B(u)$?

Some answers to the previous equations

- Bianchini-Bressan [01]: When $B(u) = Id$ and $A(u) = Df(u)$ then the sequence of solution $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, TV(\mathbb{R}))$. The sequence converges strongly in $L^1_{loc,t,x}$ to u even when the system is not conservative.

Note that $u^\varepsilon(t,x) = u(t/\varepsilon, x/\varepsilon)$ where u solves the following problem with fix viscosity but scaled initial data,

$$u_t + A(u)u_x = (B(u)_x)_x \text{ and } u(0,x) = \bar{u}(\varepsilon x). \tag{5}$$

Observe that:

$$TV(\bar{u}(\varepsilon\cdot)) = TV(\bar{u}(\cdot)).$$

Remark

We are reduced to get TV norm on u. To do this, we wish to estimate the L^1 norm of $\partial_x u$, it is a priori natural to decompose the vector u_x in the basis $(r_i(u))_{1\leq i\leq n}$:

$$u_x = \sum_{i=1}^n v_i r_i(u).$$

Differentiating (5), we obtain a system of n evolution equations:

$$v_{i,t} + (\lambda_i v_i)_x - (\mu_i v_i)_{xx} = \phi_i.$$

By maximum principle we deduce that for $t \geq \widehat{t} > 0$:

$$\|v_i(t,\cdot)\|_{L^1} \leq \|v_i(\widehat{t},\cdot)\|_{L^1} + \int_{\widehat{t}}^{t} \int_{\mathbb{R}} |\phi_i(s,x) dx ds.$$

Unfortunately in general if we consider a travelling wave solution $u(t,x) = U_i(x - \lambda t)$ with $\lim U(x)_{x \to -\infty} = U^-$ and $\lim U(x)_{x \to +\infty} = U^+$ representing a viscous i shock, we observe that:

$$\int_{\mathbb{R}} |\phi_i(t,x) dx| \neq 0.$$

Remark

This is the reason why it is important to choose a basis $(\widetilde{r}_i(u))_{1 \leq i \leq n}$ in a clever way such that $\phi_i = 0$ when we consider a viscous travelling wave. In particular we wish to have:

$$\partial_x U_i(x - \lambda t) = v_i \widetilde{r}_i(u, v_i, \lambda).$$

It is what are doing Bianchini, Bressan by using the center manifold theorem.

Remark

If each characteristic field of A belongs to the Temple class, that is, the Lax curves are straight lines and satisfying

$$Dr_i(u) \cdot r_i(u) = 0 \text{ for all } i = 1, 2, \cdots, n, \tag{6}$$

then it can be observed that $u(t,x) = U(x - \sigma t)$ with $u_x = a(s) r_i(U(s))$ forms a travelling wave when U, a, σ_i satisfying the following system of ODE,

$$\begin{aligned} U'(s) &= a(s) r_i(U(s)), \\ a'(s) &= \frac{1}{\mu_i(U(s))} \left[(\lambda_i(U(s)) - \sigma_i) a(s) - r_i(U(s)) \bullet \mu_i \right], \\ \sigma_i'(s) &= 0. \end{aligned}$$

We can show in particular in this case that setting $v_i(t,x) = a(x - \sigma t)$, we obtain:

$$\partial_t v_i + (\lambda_i(U) v_i)_x - (\mu_i(U) v_i)_{xx} = 0.$$

We set $u_x^i := l_i(u)\cdot u_x$ and we have

$$u_x = \sum_i u_x^i r_i(u). \tag{7}$$

The directional derivative of a function $g:\mathbb{R}^n\to\mathbb{R}^n$ is denoted by $\zeta\bullet g$ for some $\zeta\in\mathbb{R}^n$. More precisely,

$$\zeta\bullet g(u) = \lim_{z\to 0}\frac{g(u+z\zeta)-g(u)}{z}.$$

$$u_t+\sum_i\lambda_i u_x^i r_i = \sum_i(u_x^i B(u)r_i)_x = \sum_i(\mu_i u_x^i r_i)_x = \sum_i(\mu_i u_x^i)_x r_i+\sum_{i,j}\mu_i u_x^i u_x^j r_j\bullet r_i, \tag{8}$$

Hence, after tedious computation we can write an equation on $\partial_x u$ which gives:

$$\sum_i(u_{xt}^i+(\lambda_i u_x^i)_x-(\mu_i u_x^i)_{xx})r_i = \sum_{i,j}p_{ij}u_x^i u_x^j+\sum_{i,j}q_{ij}u_{xx}^i u_x^j+\sum_{i,j,k}s_{ijk}u_x^i u_x^j u_x^k,$$

where p_{ij}, q_{ij} and s_{ijk} are defined as follows,

$$\begin{aligned}
p_{ij} &= -\lambda_i(r_j\bullet r_i - r_i\bullet r_j),\\
q_{ij} &= 2\mu_i r_j\bullet r_i + (\mu_i-\mu_j)r_j\bullet r_i,\\
s_{ijk} &= 2(r_j\bullet\mu_i)r_k\bullet r_i+\mu_i(r_k\bullet(r_j\bullet r_i)-(r_k\bullet r_i)\bullet r_j)-(r_k\bullet\mu_j)r_j\bullet r_i.
\end{aligned}$$

Furthermore, we set $p^i_{jk} := l_i \cdot p_{jk}$, $q^i_{jk} := l_i \cdot q_{jk}$ and $s^i_{jkl} := l_i \cdot s_{jkl}$. Writing $v_i = u^i_x$, we get

$$v_{i,t} + (\lambda_i v_i)_x - (\mu_i v_i)_{xx} = \sum_{j,k} p^i_{jk} v_j v_k + \sum_{j,k} q^i_{jk} v_{j,x} v_k + \sum_{j,k,l} s^i_{jkl} v_j v_k v_l$$
$$=: \phi_i(u, v_1, \cdots, v_n).$$

We note that $p^i_{kk} = q^i_{kk} = s^i_{kkk} = 0$ for all i, k due to the assumption $r_k \bullet r_k = 0$ for all k.

Triangular system

Let us consider now the following triangular system:

$$\begin{cases} u_{1,t} + (f(u_1))_x = 0, \\ u_{2,t} + (g(u_1, u_2))_x = 0. \end{cases}$$

We consider the corresponding viscosity approximation

$$\begin{cases} u_{1,t} + (f(u_1))_x = \alpha_1 u_{1,xx}, \\ u_{2,t} + (g(u_1, u_2))_x = [(\beta(u_1, u_2) u_{1,x})_x + \alpha_2 u_{2,xx}]. \end{cases}$$

We can write (7)–(7) in the following form

$$u_t + A(u) u_x = (B(u) u_x)_x, \tag{9}$$

where A, B are defined as follows

$$A(u) = \begin{pmatrix} f'(u_1) & 0 \\ \frac{\partial g}{\partial u_1} & \frac{\partial g}{\partial u_2} \end{pmatrix} \text{ and } B(u) = \begin{pmatrix} \alpha_1 & 0 \\ \beta(u) & \alpha_2 \end{pmatrix}. \tag{10}$$

We assume that $\beta(u)$ satisfies the following condition

$$\beta(u) = (\alpha_1 - \alpha_2) \frac{\frac{\partial g}{\partial u_1}}{f'(u_1) - \frac{\partial g}{\partial u_2}}. \tag{11}$$

which corresponds to $A(u)B(u) = B(u)A(u)$. Furthermore we have:

$$r_1(u) = \begin{pmatrix} 1 \\ h(u) \end{pmatrix} \text{ and } r_2(u) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12}$$

Viscous travelling wave

We would like to decompose u_x in terms to travelling waves of (9). Let $u(t,x) = U(x - \sigma_1 t)$ be a travelling wave corresponding to 1-family. Then we have the following ordinary differential system

$$\left.\begin{aligned} \dot{u} &= v, \\ \dot{v} &= B^{-1}(u)(A(u) - \sigma)v - B^{-1}(u)(v \cdot DB(u))v, \\ \dot{\sigma} &= 0. \end{aligned}\right\} \tag{13}$$

We note that $P_1^* := (u^*, 0, \lambda_1(u^*))$ are equilibrium points. We linearize near the point P_1^* and get

$$\dot{u} = v,$$

We define $V_i, 1 \leq i \leq 2$ as follows

$$v = \sum_j V_j r_j^*, \quad V_j := l_j^* \cdot v. \tag{15}$$

The center subspace will look like

$$\mathcal{N}_1 := \{(u, v, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R};\, V_j = 0, j \neq i\}. \tag{16}$$

Note that $\dim(\mathcal{N}_1) = 4$, by Center Manifold Theorem , there exists a smooth manifold $\mathcal{M}_1 \subset \mathbb{R}^5$ which is tangent to $\mathcal{N}_1$ at P_i^*.
Furthermore, $\mathcal{M}_1$ has dimension 4 and is locally invariant under the flow of (13).
We can write

$$V_2 = \varphi_2(u, V_1, \sigma). \tag{17}$$

We can assume that φ_2 is defined on the domain

$$\mathcal{D}_1 := \{|u - u^*| < \varepsilon,\, |V_1| < \varepsilon,\, |\sigma - \lambda_1(u^*)| < \varepsilon\}. \tag{18}$$

Note that equilibrium points $(u, 0, \sigma)$ with $|u - u^*| < \varepsilon, |\sigma - \lambda_1(u^*)| < \varepsilon$ lie in $\mathcal{M}_1$ we have

$$\varphi_2(u, 0, \sigma) = 0 \text{ for all }. \tag{19}$$

Hence, we may write

$$\varphi_2(u, V_1, \sigma) = \psi_2(u, V_1, \sigma) V_1, \tag{20}$$

for some ψ_2.

It implies that:

$$v = V_1(r_1^* + \psi_2(u, V_1, \sigma)r_2^*)$$

We deduce that:

$$v = V_1\langle l_1(u), (r_1^* + \psi_2(u, V_1, \sigma)r_2^*)\rangle r_1(u) + V_1\langle l_2(u), (r_1^* + \psi_2(u, V_1, \sigma)r_2^*)r_2(u).$$

Now, we would like to make a change of coordinates $V_k \mapsto \widetilde{V}_k$ as follows

$$\widetilde{V}_k = \langle v, l_k(u)\rangle \tag{21}$$

Therefore, for any point $(u, v, \sigma) \in \mathcal{M}_1$ we can write

$$v = \widetilde{V}_1\Big(r_1(u) + \widetilde{\psi}_2(u, \widetilde{V}_1, \sigma)r_2(u)\Big) =: \widetilde{V}_1\widetilde{r}_1(u, \widetilde{V}_1, \sigma). \tag{22}$$

We note that the gradient of 1-family travelling waves can be written under the following form

$$u_x = v_1\widetilde{r}_1 \text{ where } \widetilde{r}_1 = \begin{pmatrix} 1 \\ s(u, v_1, \sigma_1) \end{pmatrix} \tag{23}$$

with

$$s(u, v_1, \sigma_1) = \widetilde{\psi}_2(u, v_1, \sigma_1) + \frac{\frac{\partial g}{\partial u_1}}{f'(u_1) - \frac{\partial g}{\partial u_2}}.$$

Furthermore we can check that:

$$\widetilde{r}_{1,\sigma} = \mathcal{O}(1)v_1, \quad \widetilde{r}_{1,\sigma\sigma} = \mathcal{O}(1)v_1, \quad \widetilde{r}_1 \bullet \widetilde{r}_{1,\sigma} = \mathcal{O}(1)v_1. \tag{24}$$

We want to have a decomposition of u_x as

$$u_x = v_1\widetilde{r}_1(u, v_1, \sigma_1) + v_2 r_2(u). \tag{25}$$

We observe that $B\widetilde{r}_1 = \alpha_1\widetilde{r}_1 + s_1 r_2$ with $s_1(u, v_1, \sigma_1)$ a function depending on (u, v_1, σ_1). Set $w_1{=}\alpha_1 v_{1,x} - \lambda_1 v_1$ the effective flux, after tedious computation we obtain that:

$$\begin{aligned}(B(u)u_x)_x - A(u)u_x &= w_1[\widetilde{r}_1 + v_1\widetilde{r}_{1,v}] + (\alpha_2 v_{2,x} - \lambda_2 v_2)r_2\\ &+ \frac{1}{\alpha_1}(w_1 + \sigma_1 v_1)(s_1 + v_1 s_{1,v})r_2 + v_1\sigma_{1,x}s_{1,\sigma}r_2\\ &+ v_1^2\sigma_1\widetilde{r}_{1,v} + \alpha_1\sigma_{1,x}v_1\widetilde{r}_{1,\sigma} + v_1 v_2 r_2 \bullet s_1 r_2 + \alpha_1 v_1 v_2 r_2 \bullet \widetilde{r}_1.\end{aligned}$$

We need now to specify the choice of σ_1. Assume that we consider a travelling wave such that $u_x = v_1\widetilde{r}_1$ then from the previous equation we should have:

$$\begin{aligned}u_t = w_1\widetilde{r}_1 &+ \frac{1}{\alpha_1}(w_1 + \sigma_1 v_1)(s_1 + v_1 s_{1,v})r_2\\ &+ (w_1 + v_1\sigma_1)v_1\widetilde{r}_{1,v}.\end{aligned}$$

Since we have $u_t = -\sigma_1 u_x$, we must choose σ_1 such that $\sigma_1 = -\frac{w_1}{v_1}$.

Since σ_1 must live in a neighborhood of $\lambda_1(u^*)$, we set

$$\sigma_1 = \lambda_1(u^*) + \theta(-\frac{w_1}{v_1} - \lambda_1(u^*)),$$

with

$$\theta(s) = \begin{cases} s & \text{if } |s| \leq \frac{\delta_1}{2}, \\ 0 & \text{if } |s| \geq \delta_1, \end{cases} \qquad |\theta'| \leq 1 \text{ and } |\theta''| \leq 4/\delta_1. \tag{26}$$

Let us assume now that we consider a solution u_x of our system such that:

$$u_x = v_1 \tilde{r}_1(u, v_1, \sigma_1) + v_2 r_2(u). \tag{27}$$

If we consider the first coordinate of u_x and due to the form of $\tilde{r_1}$, we observe that $v_1 = u_{1,x}$. After computations, we can show now that (v_1, v_2)

$$\begin{cases} v_{1,t} + (\lambda_1 v_1)_x - \alpha_1 v_{1,xx} = 0, \\ v_{2,t} + (\lambda_2 v_2)_x - \alpha_2 v_{2,xx} = \phi_2, \end{cases} \tag{28}$$

with:

$$\begin{aligned} \phi_2 =& O(1)[v_{1,x}(w_1 + \sigma_1 v_1)| \quad \text{wrong speed} \\ &+ O(1)|w_{1,x} v_1 - v_{1,x} w_1| \quad \text{change in strength} \\ &+ O(1)|v_1[v_1(\frac{w_1}{v_1})_x^2]|\chi_{\{x, |\frac{w_1}{v_1}| \leq 3\delta_1\}} \quad \text{change in speed} \\ &+ O(1)[|v_1 v_2| + |v_{1,x} v_2|] \quad \text{transversal interactions} \\ &+ O(1)[| \boxed{\frac{1}{\alpha_1}((w_1 + \sigma_1 v_1)(s_1 + v_1 s_{1,v}))_x} + \boxed{(v_1 \sigma_{1,x} s_{1,\sigma})_x} |] \end{aligned}$$

Theorem

Consider the Cauchy problem hyperbolic system with viscosity,

$$u_t + A(u)u_x = \varepsilon(B(u)u_x)_x, \quad u(0,x) = \bar{u}(x). \tag{29}$$

There exists $L_1, L_2, L_3 > 0$ *and* $\delta_0 > 0$ *such that the following holds. If* $\bar{u}$ *satisfies*

$$TV(\bar{u}) \leq \delta_0 \text{ and } \lim_{x\mathbb{R}-} \bar{u}(x) \in K, \tag{30}$$

for some compact set $K \subset \mathcal{U}$ *then there exists unique solution* u^ε *to the Cauchy problem* (29) *and it satisfies the following properties*

$$TV(u^\varepsilon(t)) \leq L_1 TV(\bar{u}), \tag{31}$$

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} \leq L_2 \|\bar{u} - \bar{v}\|_{L^1}, \tag{32}$$

$$\|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1} \leq L_3 \left(|t-s| + \sqrt{\varepsilon}|\sqrt{t} - \sqrt{s}|\right), \tag{33}$$

where v^ε *is the unique solution corresponding to* $\bar{v}$ *satisfying* (30). *Furthermore, when* $A = Df$ *for some* $f \in C^1$, *as* $\varepsilon \to 0$ *(up to a subsequence),* $u^\varepsilon \to u$ *in* L^1_{loc} *with* u *a solution to hyperbolic system* (5).

Theorem

Then for every compact set $K \subset \mathcal{U}$ there exist L_1, L_2, δ_0, a closed domain $\mathcal{D} \subset L^1_{loc}(\mathbb{R})$ and a semigroup $S : [0, \infty) \times \mathcal{D} \to \mathcal{D}$ satisfying the following properties.

1. *Every function $\bar{u}$ verifying (30) belongs to $\mathcal{D}$.*
2. *For any $\bar{u}, \bar{v} \in \mathcal{D}$ with $\bar{u} - \bar{v} \in L^1$,*

$$\|S_{t_1}(\bar{u}) - S_{t_2}(\bar{v})\|_{L^1} \leq L\|\bar{u} - \bar{v}\|_{L^1} + L'|t_1 - t_2| \text{ for any } t_1, t_2 \geq 0, \tag{34}$$

   for some constants L, L' which are depending only on $\mathcal{D}$.
3. *For any piece-wise constant initial data $\bar{u} \in \mathcal{D}$ there exists $\tau > 0$ such that the following holds. For $t \in [0, \tau]$, S_t coincides with the solution constructed by gluing the Riemann problem solutions arising at each jump point.*
4. *For each $\bar{u} \in \mathcal{D}$, $t \mapsto S_t(\bar{u})$ is the unique limit of the sequence $u^{\varepsilon_k}(t, \cdot)$ in L^1_{loc} for any $\varepsilon_k \to 0$ where $u^{\varepsilon_k}(t, \cdot)$ solves (1) with initial data $\bar{u}$.*

Remark

Let S^I_t be the semigroup constructed by Bianchini-Bressan for (5). Due to the characterization S3, we conclude that the semigroup S^B_t constructed as above coincides with S^I_t.

Parabolic estimates

Proposition

Let u be a solution to the equation (5) satisfying

$$\|u_x(t,\cdot)\|_{L^1} \leq \delta_0 \textit{ for all } t \in [0,\widehat{t}] \quad \textit{where } \widehat{t} := \left(\frac{1}{C\delta_0}\right)^2, \tag{35}$$

for some $\delta_0 < 1$ and $C > 0$. Then we have

$$\begin{aligned} \|u_{xx}(t,\cdot)\|_{L^1} &\leq \frac{2\kappa\kappa_1^2\kappa_P^2\delta_0}{\sqrt{t}} \\ \|u_{xxx}(t,\cdot)\|_{L^1} &\leq \frac{5\kappa\kappa_1^2\kappa_P^2\delta_0}{\sqrt{t}} \\ \|u_{xxx}(t,\cdot)\|_{L^\infty} &\leq \frac{16\kappa\kappa_1^2\kappa_P^2\delta_0}{\sqrt{t}} \end{aligned} \tag{36}$$

First we have:

$$(u_x)_t + A(u)u_{xx} = B(u)u_{xxx} + (u_x \bullet B(u)u_x)_x - u_x \bullet A(u)u_x. \tag{37}$$

We would like to diagonalize the system, making a change of variable $v = P(u)u_x$ we get:

$$\begin{aligned} v_t + A_1(u)v_x = {} & B_1(u)v_{xx} - B_1((P^{-1}(u)v) \bullet PP^{-1}v)_x - B_1[(P^{-1}v) \bullet P(P^{-1}v)_x] \\ & + u_t \bullet P(u)P^{-1}v + A_1(u)(P^{-1}(u)v \bullet P(u)P^{-1}v) \\ & + P(P^{-1}v \bullet B(u)P^{-1}v)_x - P(P^{-1}v) \bullet A(u)P^{-1}v. \end{aligned}$$

with $A_1 = PAP^{-1}$ and $B_1 = \text{diag}(\mu_1(u), \cdots, \mu_n(u)) = PBP^{-1}$. Next, we do a change of variable $v \mapsto \widetilde{v}$ such that $v(t,x) = (\widetilde{v}_i(t, X_i(t,x)))$ where $(X_i)_x = \frac{1}{\sqrt{\mu_i(u)}}$. Then we have

$$\begin{aligned} \widetilde{v}_t + A_2^* \widetilde{v}_x = {} & \widetilde{v}_{xx} + \mathcal{T}([A_2^* - A_1^* B_1^{-1/2}(u)]\widetilde{v}_x + B_1(P^{-1}v) \bullet B_1^{-1/2}\widetilde{v}_x) \\ & - \mathcal{T}(\widetilde{v}_{i,x} X_{i,t}) + \mathcal{T}(\mathcal{R}), \end{aligned} \tag{38}$$

where $A_2^* = A_1^* B_1^{-1/2}(u^*)$ with:

$$\mathcal{T}(f)_i(x) = f_i(X_i(x)) \text{ where } X_i(x) = \int_0^x \frac{1}{\sqrt{\mu_i(u(z))}} dz. \tag{39}$$

We consider G as the fundamental solution of the following parabolic equation

$$w_t + A_2^* w_x = w_{xx}, \tag{40}$$

where $A_2^* = A_1^* B_1^{-1/2}(u^*)$. The function G satisfies the following estimates

$$\|G(t,\cdot)\|_{L^1} \leq \kappa, \quad \|G_x(t,\cdot)\|_{L^1} \leq \frac{\kappa}{\sqrt{t}}, \quad \|G(t,\cdot)\|_{L^1} \leq \frac{\kappa}{t}, \tag{41}$$

We argue by contradiction. To this end, first we assume that the conclusion does not hold. Due to the assumption of smoothness of initial data, solution is smooth up to a small time and due to the continuity we can assume that there exists a time t^* such that (47) holds for $t \in [0, t^*]$ and equality attains at $t = t^*$. We can write for $t \in [0, t^*]$:

$$\begin{aligned}\widetilde{v}_x = G_x(t/2) \star \widetilde{v}(t/2) + \int_{t/2}^{t} G_x(t-s) \star \Big\{ \mathcal{T}([A_2^* - A_1^* B_1^{-1/2}(u)]\widetilde{v}_x \\ + B_1(P^{-1}v) \bullet B_1^{-1/2}\widetilde{v}_x) - \mathcal{T}(\widetilde{v}_{i,x} X_{i,t}) + \mathcal{T}(\mathcal{R}) \Big\}\, ds.\end{aligned}$$

Furthermore, we observe that

$$\|\widetilde{v}\|_{L^1} \leq \kappa_P \kappa_1 \|u_x\|_{L^1}, \tag{42}$$
$$\|\widetilde{v}_x\|_{L^1} \leq \kappa_P \kappa_1 \|u_{xx}\|_{L^1}. \tag{43}$$

We get:

$$\begin{aligned}
\|\widetilde{v}_x(t)\|_{L^1} &\leq \|G_x(t/2)\|_{L^1}\|\widetilde{v}(t/2)\|_{L^1} + \int_{t/2}^{t} \|G_x(t-s)\|_{L^1}\|\mathcal{T}([A_2^* - A_1^* B_1^{-1/2}(u)]\widetilde{v}_x)\|_{L^1}\, ds \\
&+ \int_{t/2}^{t} \|G_x(t-s)\|_{L^1}\|\mathcal{T}(B_1(P^{-1}v)\bullet B_1^{-1/2}\widetilde{v}_x) - \mathcal{T}(\widetilde{v}_{i,x}X_{i,t}) + \mathcal{T}(\mathcal{R})\|_{L^1}\, ds \\
&\leq \frac{2\kappa\delta_0}{\sqrt{t}} + \int_{t/2}^{t} \frac{\kappa\kappa_1}{\sqrt{t-s}}\|[A_2^* - A_1^* B_1^{-1/2}(u)]\widetilde{v}_x\|_{L^1}\, ds \\
&+ \int_{t/2}^{t} \frac{\kappa\kappa_1}{\sqrt{t-s}}\|B_1(P^{-1}v)\bullet B_1^{-1/2}\widetilde{v}_x - (\widetilde{v}_{i,x}X_{i,t}) + \mathcal{R}\|_{L^1}\, ds.
\end{aligned}$$

We note that

$$\|[A_2^* - A_1^* B_1^{-1/2}(u)]\widetilde{v}_x\|_{L^1} \leq \kappa_A \kappa_B \|u - u^*\|_{L^\infty} \|\widetilde{v}_x\|_{L^1}$$
$$\leq \kappa_A \kappa_B \kappa_P \kappa_1 \|u - u^*\|_{L^\infty} \|u_{xx}\|_{L^1},$$

Therefore, we get

$$\|\widetilde{v}_x(t)\|_{L^1} \leq \frac{\sqrt{2}\kappa\kappa_1\kappa_P\delta_0}{\sqrt{t}} + 600\kappa_1^6\kappa^3\kappa_A\kappa_B^7\kappa_P^{12} \int_{t/2}^{t} \frac{1}{\sqrt{t-s}} \left[\frac{\delta_0^2}{s} + \frac{\delta_0^2}{\sqrt{s}}\right] ds$$
$$< \frac{2\kappa\kappa_1\kappa_P\delta_0}{\sqrt{t}},$$

which implies

$$\|u_{xx}(t^*, \cdot)\|_{L^1} < \frac{2\kappa\kappa_1^2\kappa_P^2\delta_0}{\sqrt{t^*}}. \tag{44}$$

This contradicts the assumption that equality holds in (47) at $t = t^*$.

Proposition

Let $\bar{u}$ satisfying

$$TV(\bar{u}) \leq \frac{\delta_0}{4\kappa}.$$

Then u are well-defined on $[0, \widehat{t}]$ where $\widehat{t}$ is defined as in (46). Moreover, we have

$$\|u_x(t)\|_{L^1} \leq \frac{\delta_0}{2} \text{ for } t \in [0, \widehat{t}]. \tag{45}$$

Suppose that there exists a time $\tau < \widehat{t}$ such that $\|u_x(\tau)\|_{L^1} = \frac{\delta_0}{2}$ and $\|u_x(t)\|_{L^1} < \frac{\delta_0}{2}$ for all $t \in [0, \tau]$. We can write

$$u_t + A(u^*)u_x = B(u^*)u_{xx} + (B(u) - B(u^*))u_{xx} + (A(u^*) - A(u))u_x + u_x \bullet Bu_x.$$

Therefore,

$$\|u_x(\tau)\|_{L^1} \leq \kappa\|u_{0,x}\|_{L^1} + \int_0^\tau \frac{2\kappa\kappa_B}{\sqrt{\tau - s}}\|u_x\|_{L^1}\|u_{xx}\|_{L^1}\, ds + \int_0^\tau \frac{\kappa\kappa_A}{\sqrt{\tau - s}}\|u_x\|_{L^1}^2.$$

By Proposition 2.1 we get

$$\|u_{xx}(t,\cdot)\|_{L^1} \leq \frac{2\kappa\kappa_1^2\kappa_P^2\delta_0}{\sqrt{t}} \text{ for } t \in [0, \tau].$$

Hence, by the choice of δ_0

$$\|u_x(\tau)\|_{L^1} \leq \frac{\delta_0}{4} + 2\int_0^\tau \frac{\kappa}{\sqrt{\tau - s}} \frac{2\kappa\kappa_1^2\kappa_P^2\delta_0}{\sqrt{s}} \frac{\kappa_B\delta_0}{2}\, ds + \int_0^\tau \frac{\kappa}{\sqrt{\tau - s}} \frac{\kappa_A\delta_0^2}{4}\, ds$$
$$< \frac{\delta_0}{2}.$$

Proposition

Let $T > \widehat{t}$ and u be a solution to the equation (5) satisfying

$$\|u_x(t,\cdot)\|_{L^1} \leq \delta_0 \text{ for all } t \in [0,T] \tag{46}$$

for some $\delta_0 < 1$ and $C > 0$. Then we have for $t \in [\widehat{t}, T]$

$$\begin{aligned} \|u_{xx}(t,\cdot)\|_{L^1} &= O(1)\delta_0^2 \\ \|u_{xxx}(t,\cdot)\|_{L^1} &= O(1)\delta_0^3 \\ \|u_{xxx}(t,\cdot)\|_{L^\infty} &= O(1)\delta_0^4. \end{aligned} \tag{47}$$

Interaction estimates

Lemma

Let $z, z^{\#}$ *be solutions of the two independent scalar equations,*

$$z_t + (\lambda(t,x)z)_x - (\mu z)_{xx} = \varphi(t,x), \tag{48}$$

$$z_t^{\#} + (\lambda^{\#}(t,x)z^{\#})_x - (\mu^{\#} z^{\#})_{xx} = \varphi^{\#}(t,x), \tag{49}$$

which is valid for $t \in [0,T]$. *We assume that* $\inf_{t,x} \lambda^{\#}(t,x) - \sup_{t,x} \lambda(t,x) \geq c > 0$ *and* $\|(\mu, \mu^{\#})\|_{L^\infty} < \infty$, $\mu, \mu^{\#} \geq c_0 > 0$. *Then we have*

$$\int_0^T \int_{\mathbb{R}} |z(t,x)||z^{\#}(t,x)|\, dxdt \leq \frac{1}{c} E_1 E_2, \tag{50}$$

$$E_1 := \int_{\mathbb{R}} |z(0,x)|\, dx + \int_0^T \int_{\mathbb{R}} |\varphi(t,x)|\, dxdt, \tag{51}$$

$$E_2 := \int_{\mathbb{R}} |z^{\#}(0,x)|\, dx + \int_0^T \int_{\mathbb{R}} |\varphi^{\#}(t,x)|\, dxdt. \tag{52}$$

Set $c_1 := \|\mu, \mu^{\#}\|_{L^\infty}$. Let $z, z^{\#}$ be the solution to (48), (49) with $\varphi = \varphi^{\#} = 0$. Consider

$$Q(z, z^{\#}) := \int\int K(x-y)|z(x)|\,|z^{\#}(y)|\,dxdy, \tag{53}$$

where K is defined as follows

$$K(s) := \begin{cases} 1/c & \text{if } s \geq 0, \\ 1/ce^{\frac{cs}{2c_1}} & \text{if } s < 0. \end{cases} \tag{54}$$

Now, we can calculate using the fact that $cK' - 2c_1 K''$ is precisely the Dirac masses

$$\begin{aligned}
\frac{d}{dt}Q(z(t), z^{\#}(t)) &= \int\int K(x-y)[sgn(z(x))z_t(x)|z^{\#}(y)| + sgn(z^{\#}(y))z^{\#}_t(y)|z(x)|]\,dxdy \\
&= \int\int K(x-y)\Big[sgn(z(x))((\mu(x)z(x))_{xx} - (\lambda z(x))_x)|z^{\#}(y)| \\
&\quad + sgn(z^{\#}(y))((\mu^{\#}(y)z^{\#}(y))_{yy} - (\lambda^{\#} z^{\#}(y))_y)|z(x)|\Big]\,dxdy \\
&\leq -\int\int (cK'(x-y) - 2c_1 K''(x-y))|z(x)||z^{\#}(y)|\,dxdy \\
&\leq -\int |z(x)||z^{\#}(x)|\,dx.
\end{aligned}$$

$$\int_0^T \int_{\mathbb{R}} |z(t,x)|\,|z^{\#}(t,x)|\,dxdt \leq Q(z(0), z^{\#}(0)) \leq \frac{1}{c}\|z(0)\|_{L^1}\|z^{\#}(0)\|_{L^1}. \tag{55}$$

Now, we consider $z, z^{\#}$ as solutions of (48), (49) respectively when φ and $\varphi^{\#}$ may not be identically 0. Using the representation of the solution in terms of $\Gamma, \Gamma^{\#}$ be the fundamental solutions corresponding to the homogeneous system of (48)–(49) we can conclude in a similar way. Indeed we can write

$$z(t,x) = \int_{\mathbb{R}} \Gamma(t,x,0,y) z(0,y)\, dy + \int_0^t \int_{\mathbb{R}} \Gamma(t,x,s,y)\varphi(s,y)\, dyds. \tag{56}$$

Lemma

Let $z, z^{\#}$ be solutions of (48), (49) respectively and we assume that

$$\int_0^T \int_{\mathbb{R}} |\varphi(t,x)| dxdt \le \delta_0, \qquad \int_0^T \int_{\mathbb{R}} |\varphi^{\#}(t,x)| dxdt \le \delta_0, \tag{57}$$

$$\|z(t)\|_{L^1},\ \|z^{\#}(t)\|_{L^1} \le \delta_0, \qquad \|z_x(t)\|_{L^1},\ \|z^{\#}(t)\|_{L^\infty} \le C_* \delta_0^2, \tag{58}$$

$$\|\lambda_x(t)\|_{L^\infty},\ \|\lambda_x(t)\|_{L^1} \le C_* \delta, \qquad \lim_{x\to-\infty} \lambda(t,x) = 0, \tag{59}$$

for all $t \in [0,T]$. Then we have

$$\int_0^T \int_{\mathbb{R}} |z_x(t,x)||z^{\#}(t,x)|\, dxdt = \mathcal{O}(1)\delta_0^2. \tag{60}$$

Lemma

Let $z, z^{\#}$ *be solutions of* (48), (49) *respectively and we assume that*

$$\int_0^T \int_{\mathbb{R}} |\varphi(t,x)| dx dt \leq \delta_0, \qquad \int_0^T \int_{\mathbb{R}} |\varphi^{\#}(t,x)| dx dt \leq \delta_0, \tag{61}$$

$$\|z(t)\|_{L^1}, \|z^{\#}(t)\|_{L^1} \leq \delta_0, \qquad \|z_x(t)\|_{L^1}, \|z^{\#}(t)\|_{L^\infty} \leq C_* \delta_0^2, \tag{62}$$

$$\|\lambda_x(t)\|_{L^\infty}, \|\lambda_x(t)\|_{L^1} \leq C_* \delta, \qquad \lim_{x \to -\infty} \lambda(t,x) = 0, \tag{63}$$

for all $t \in [0,T]$. *Then we have*

$$\int_0^T \int_{\mathbb{R}} |z_x(t,x)||z^{\#}(t,x)| \, dx dt = \mathcal{O}(1) \delta_0^2. \tag{64}$$

Let v, w be two scalar functions satisfying:

$$\begin{cases} v_t + (\lambda(t,x)v)_x - (\mu v)_{xx} & = \varphi(t,x), \\ w_t + (\lambda(t,x)w)_x - (\mu w)_{xx} & = \varphi^{\#}(t,x), \end{cases}$$

Considering the functionnals as:

$$\mathcal{A}(t) = \frac{1}{2}\int\int_{x<y} |v(t,x)w(t,y) - v(t,y)w(t,x)|dxdy,$$
$$\mathcal{L}(t) = \int \sqrt{v^2(t,x) + w^2(t,x)}dx$$

we have then the following Lemmas.

Lemma

The previous functionals satisfies:

$$\frac{d}{dt}\mathcal{A}(t) + \int |v_x(t,x)w(t,x) - w_x(t,x)v(t,x)|dx$$
$$\|v(t)\|_{L^1}\|\varphi^{\#}(t)\|_{L^1} + \|w(t)\|_{L^1}\|\varphi(t)\|_{L^1} \qquad (65)$$
$$\frac{d}{dt}\mathcal{L}(t) \leq -C(\delta_1)\int_{|\frac{w}{v}|\leq 3\delta_1} [v(t)|[\frac{w(t)}{v(t)})_x|^2 dx + \|\varphi^{\#}(t)\|_{L^1} + \|\varphi(t)\|_{L^1}.$$

TV Bounds

Let us consider an initial data satisfying $TV(\bar{u}) \leq \frac{\delta_0}{8\sqrt{n}}$ and $\lim_{x \to -\infty} u(x) = u^* \in K$. Then by applying Proposition 2.2, we obtain

$$\|u_x(\widehat{t})\|_{L^1(\mathbb{R})} \leq \frac{\delta_0}{4\sqrt{n}}, \tag{66}$$

where $\widehat{t}$ is defined as in (46). To get the total variation bound in $(\widehat{t}, \infty)$ we argue by contradiction as in Bianchini-Bressan. Let T be defined as follows

$$T := \sup \left\{ \tau; \sum_i \int_{\widehat{t}}^{\tau} \int_{\mathbb{R}} |\phi_i(t,x)| \, dxdt \leq \frac{\delta_0}{2} \right\}. \tag{67}$$

It $T < +\infty$, we get a contradiction as follows. From (67), we have

$$\|u_x(t)\|_{L^1} \leq 2\sqrt{n}\|u_x(\widehat{t})\|_{L^1} + \frac{\delta_0}{2} \leq \delta_0 \text{ for all } t \in [\widehat{t}, T]. \tag{68}$$

By applying 6, we get

$$\int_{\widehat{t}}^{\tau} \int_{\mathbb{R}} |\sum_{j,k} q^i_{jk} v_{j,x} v_k| \, dxdt = \mathcal{O}(1)\delta_0^2 < \frac{\delta_0}{2} \tag{69}$$

for sufficiently small $\delta_0 > 0$.

Indeed we observe that $\|v_{j,x}(t)\|_{L^1}$, $\|v_i^{\#}(t)\|_{L^\infty} \leq C_* \delta_0^2$ are satisfied using the proposition 2.3.

How to deal with the new terms in the case of the triangular system?

We recall that in ϕ_2 we have new terms of the form:

$$(v_1 \sigma_{1,x} s_{1,\sigma})_x.$$

Since $s_{1,\sigma} = O(1)v_1$ and $\sigma_1 = \lambda_1(u^*) + \theta(-\frac{w_1}{v_1} - \lambda_1(u^*))$, we have to deal with a term of the form:

$$w_{1,xx} v_1 - v_{1,xx} w_1.$$

First of all we can observe that:

$$w_{1,t} + (\lambda_1(u) w_1)_x - \alpha_1 w_{1,xx} = 0. \tag{70}$$

Furthermore we have $\lambda_1(u) = f'(u_1)$, we introduce now a new variable

$$z_1 = \alpha_1 w_{1,x} - \lambda_1(u) w_1.$$

We can check that this new unknown satisfies:

$$z_{1,t} = \alpha_1 z_{1,xx} - (\lambda_1 z_1)_x + \lambda_1'(w_{1,x} v_1 - w_1 v_{1,x}).$$

We observe now that:

$$z_{1,x}v_1 - v_{1,x}z_1 = w_{1,xx}v_1 - w_1 v_{1,xx} + 2\lambda_1(w_1 v_{1,x} - w_{1,x}v_1). \tag{71}$$

Using now (65), we deduce that:

$$\begin{aligned}
&\int_{\widehat{t}}^{\tau}\int_{\mathbb{R}} |w_{1,xx}v_1 - w_1 v_{1,xx}|\, dxdt \\
&\leq \int_{\widehat{t}}^{\tau}\int_{\mathbb{R}} |z_{1,x}v_1 - v_{1,x}z_1|\, dxdt + 2\|\lambda_1\|_{L^\infty}\int_{\widehat{t}}^{\tau}\int_{\mathbb{R}} |w_{1,x}v_1 - v_{1,x}w_1|\, dxdt \\
&\leq \|z_1(\widehat{t})\|_{L^1}\|v_1(\widehat{t})\|_{L^1} + \|v_1\|_{L^\infty((\widehat{t},\tau);L^\infty(\mathbb{R}))}\|\lambda_1'(w_{1,x}v_1 - w_1 v_{1,x})\|_{L^1([\widehat{t},\tau]\times\mathbb{R})} \\
&+ 2\|\lambda_1\|_{L^\infty}\|w_{1,x}v_1 - v_{1,x}w_1\|_{L^1([\widehat{t},\tau]\times\mathbb{R})}.
\end{aligned}$$

Using again (65), we conclude since:

$$\|w_{1,x}v_1 - v_{1,x}w_1\|_{L^1([\widehat{t},\tau]\times\mathbb{R})} = 0(1)\delta_0^2.$$

Hence, T is not the supremum defined as in (67). Hence, $\int_{\widehat{t}}^{\tau}\int_{\mathbb{R}} |\phi_i(t,x)|\, dxdt \leq \frac{\delta_0}{2}$ for all $t > \widehat{t}$. Subsequently, we obtain for all $t \geq 0$:

$$\|u_x(t)\|_{L^1} \leq \delta_0.$$

Stability Estimates

Let $\bar{u}^\theta$ be the initial defined as follows

$$\bar{u}^\theta := \theta\bar{u} + (1-\theta)\bar{v} \text{ for some } \bar{u}, \bar{v} \in \mathcal{D}. \tag{72}$$

Let u^θ be the solution associated to initial data $\bar{u}^\theta$. Then taking derivative w.r.t.

We can now write after tedious computations

$$\sum_i (h_{i,t} + (\lambda_i h_i)_x - (\mu_i h_i)_{xx}) r_i$$
$$= \sum_{i,j} \widehat{p}_{ij} h_i v_j + \sum_{i,j,k} \widehat{q}_{ijk} h_i v_j v_k + \sum_{i,j} \widehat{s}_{ij} h_{i,x} v_j + \sum_{i,j} \widehat{w}_{ij} h_i v_{j,x},$$

where $\widehat{p}_{ij}, \widehat{q}_{ijk}, \widehat{s}_{ij}, \widehat{w}_{ij}$ are defined as follows

$$\begin{aligned}
\widehat{p}_{ij} &= (\lambda_j - \lambda_i) r_j \bullet r_i + r_j \bullet A r_i - r_i \bullet A r_j, \\
\widehat{q}_{ijk} &= -(r_k \bullet \mu_j) r_j \bullet r_i - \mu_j (r_k \bullet r_j) \bullet r_i + 2(r_k \bullet \mu_i) r_j \bullet r_i + \mu_i (r_k \bullet (r_j \bullet r_i)) \\
&\quad + (r_k \bullet r_i) \bullet B r_j - (r_k \bullet r_j) \bullet B r_i + r_i \bullet B(r_k \bullet r_j) - r_j \bullet B(r_k \bullet r_i) \\
&\quad + (r_j \otimes r_i) : D^2 B r_k - (r_j \otimes r_k) : D^2 B r_i, \\
\widehat{s}_{ij} &= 2\mu_i (r_j \bullet r_i) + r_i \bullet B r_j - r_j \bullet B r_i, \\
\widehat{w}_{ij} &= (\mu_i - \mu_j) r_j \bullet r_i - r_j \bullet B r_i + r_i \bullet B r_j.
\end{aligned}$$

By using interaction estimates (Lemma 3 and 6) we can obtain

$$\|h(t)\|_{L^1} \leq \frac{\|h(\widehat{t})\|_{L^1}}{2} \text{ for all } t > \widehat{t}. \tag{73}$$

Combining the above inequality with Proposition 2.2, we have

$$\|h(t)\|_{L^1} \leq L_3 \|h_0\|_{L^1} \leq L_3 \|\bar{u} - \bar{v}\|_{L^1} \text{ for all } t > 0. \tag{74}$$

To translate this result to the L^1 stability estimate of two viscosity solutions u, v we use the homotopy method as in Bianchini-Bressan. Let u^θ be the solution corresponding to the initial data $\theta\bar{u} + (1-\theta)\bar{v}$. Then let h be defined as $h^\theta := \frac{du^\theta}{d\theta}$. Then for all $t > 0$ we have:

$$\|u(t) - v(t)\|_{L^1} \leq \int_0^1 \|\frac{du^\theta(t)}{d\theta}\|_{L^1}\, d\theta \leq L_3\|\bar{u} - \bar{v}\|_{L^1}. \tag{75}$$

Stability Estimates

As claimed in Theorem 1 we want to prove vanishing viscosity limit as $\varepsilon\mathbb{R}0$ for the following Cauchy problem

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon = \varepsilon(B(u^\varepsilon)_x)_x \text{ and } u^\varepsilon(0,x) = \bar{u}(x). \tag{76}$$

Note that $u^\varepsilon(t,x) = u(t/\varepsilon, x/\varepsilon)$ where u solves the following problem with fix viscosity but scaled initial data,

$$u_t + A(u)u_x = (B(u)_x)_x \text{ and } u(0,x) = \bar{u}(\varepsilon x). \tag{77}$$

Observe that

$$TV(\bar{u}(\varepsilon\cdot)) = TV(\bar{u}(\cdot)).$$

Therefore, we obtain

$$TV(u^\varepsilon(t)) \leq L_1 TV(\bar{u}), \tag{78}$$

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} = \varepsilon\|u(t) - v(t)\|_{L^1} \leq L_2\|\bar{u} - \bar{v}\|_{L^1}, \tag{79}$$

$$\|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1} \leq L_3 \left(|t-s| + \sqrt{\varepsilon}|\sqrt{t} - \sqrt{s}|\right), \tag{80}$$

$$|u^\varepsilon(t,x) - v^\varepsilon(t,x)| \leq \alpha_1\|\bar{u} - \bar{v}\|_{L^\infty} \left(e^{\frac{c_1}{\varepsilon}(\beta_1 t-(x-a))} + e^{\frac{c_1}{\varepsilon}(\beta_1 t+(x-b))}\right). \tag{81}$$

The convergence of u^ε as $\varepsilon \to 0$ follows from a standard argument with an application of Helly's theorem and the L^1 continuity (80).
For $\delta_0 > 0$ and compact set $K \subset \mathcal{U}$, we consider

$$\mathcal{D}_0 := \{u : \mathbb{R} \to \mathbb{R}^n;\ u(-\infty) \in K \text{ and } TV(u) \leq \delta_0\} \tag{82}$$

By considering a smaller domain $\mathcal{D} \subset \mathcal{D}_0$ which is positively invariant, we can set $S : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$. From (80) and (81) we conclude the time continuity and continuous dependence on initial data for S_t.

Uniqueness of the semi group

We first recall the Riemann solver for equation of hyperbolic systems

$$u_t + A(u)u_x = 0, \text{ with } u(0,x) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \tag{83}$$

with $|u_l - u_r|$ is small enough. We consider the i-th rarefaction curve $\sigma \mapsto \mathcal{R}_i(\sigma; u_-)$ starting from $u_- \in \Omega$ which satisfies

$$\frac{d}{d\sigma}\mathcal{R}_i(\sigma; u_-) = r_i(\mathcal{R}_i(\sigma; u_-)) \text{ with } \mathcal{R}_i(0; u_-) = u_-. \tag{84}$$

By using implicit function theorem and with the help of strict hyperbolicity, there exist $\bar{\lambda}_1 < \cdots < \bar{\lambda}_{n-1}$, $\{\sigma_i\}_{i=1}^n$ and $\{w_i\}_{i=0}^n$ such that

$$w_0 = u_l,\, w_n = u_r \text{ and } w_i = \mathcal{R}_i(\sigma; w_{i-1}) \text{ for } i = 1, 2, \cdots, n. \tag{85}$$

Moreover, $\lambda_i(\mathcal{R}_i(\theta\sigma_i; w_{i-1})) \in (\bar{\lambda}_{i-1}, \bar{\lambda}_i)$ for $\theta \in [0,1]$ and $1 \leq i \leq n$ where $\bar{\lambda}_0 := -\infty$ and $\bar{\lambda}_n := +\infty$. Let us consider scalar flux F_i corresponding to i-characteristics defined as follows

$$F_i(\omega) := \int_0^\omega \lambda_i(\mathcal{R}_i(s; w_{i-1}))\, ds. \tag{86}$$

Let z_i be the unique entropy solution the following Cauchy problem for scalar conservation laws,

$$z_{i,t} + F_i(z_i)_x = 0, \tag{87}$$

$$z_i(0,x) = \begin{cases} 0 & \text{if } x < 0, \\ \sigma_i & \text{if } x > 0. \end{cases} \tag{88}$$

Now, we can describe the solution to (83) as follows,

$$u(t,x) = \mathcal{R}_i(z_i(t,x); w_{i-1}) \text{ for } \frac{x}{t} \in [\bar{\lambda}_{i-1}, \bar{\lambda}_i] \text{ for all } i = 1, \cdots, n. \tag{89}$$

We first consider the Riemann data where u_-, u_+ both lie on i-rarefaction curve. Since the rarefaction curves are straight lines we can write

$$\bar{u}(x) = u^* + \bar{z}(x) r_i(u^*) \text{ where } u^* = u(-\infty).$$

Consider the flux F_i defined as in (86). Then we note that since the solution u^ε is satisfying (1), we obtain

$$z_t^\varepsilon + F(z^\varepsilon)_x = \varepsilon(\mu_i(u^\varepsilon) z_x^\varepsilon)_x \text{ where } u^\varepsilon = u^* + z^\varepsilon r_i(u^*) \text{ and } z(0,x) = \bar{z}(x). \tag{90}$$

Due to uniform parabolicity, global solution z^ε exists and z^ε converges to entropy solution z of (87).

Hence, the limit u can be written $u(t,x) = u^* + z(t,x) r_i(u^*)$, in other words, u^ε converges to a solution of the hyperbolic system, u defined as in (89).

Lemma (Finite speed of propagation)

Let $\bar{u}, \bar{v} \in \mathcal{D}$. Then there exists $\beta_1 > 0$ such that the following holds for $a, b \in \mathbb{R}$,

$$\int_a^b |S_t(\bar{u}) - S_t(\bar{v})|\, dx \leq L_4 \int_{a-\beta_1 t}^{b+\beta_1 t} |\bar{u} - \bar{v}|\, dx. \tag{91}$$

We consider an initial data which is perturbation of a Riemann data $\bar{u}_{Rie} = u_- \chi_{(-,0)} + u_+ \chi_{(0,)}$ defined as follows

$$\bar{u}(x) := \begin{cases} u_- & \text{if } x < \delta, \\ w_i & \text{if } i\delta < x < (i+1)\delta, \text{ with } 1 \leq i \leq n-1, \\ u_+ & \text{if } x > n\delta. \end{cases} \tag{92}$$

Due to finite speed of propagation, up to a small time t_0, the waves do not interact with each other and the limit solution when ε goes to 0 can be written as

$$u^\delta(t,x) = \mathcal{R}_i(z_i(t, x - i\delta); w_{i-1}) \text{ for } x \in [i\delta + \widehat{\lambda} t, (i+1)\delta - \widehat{\lambda} t] \text{ when } t \in [0, t_0]. \tag{93}$$

Now, by sending $\delta \to 0$, we can obtain that the viscosity solution u^ε converges to solution of te hyperbolic system for Riemann data. Since a Lipschitz continuous semigroup is determined by the local in time behavior for piecewise constant data, this characterizes the limit function u. Moreover, it also says that for any subsequence $\varepsilon_k \to 0$, u^{ε_k} converges to the same limit. This completes the proof of Theorem 2.

MERCI POUR VOTRE ATTENTION!!!