Asymptotics for the strong solutions of the strongly stratified Boussinesq system with ill-prepared initial data

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Incompressible Navier-Stokes system

(NS)

$$\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v = -\nabla p, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ \mathrm{div}\, v = 0, \\ v_{|t=0} = v_0. \end{cases}$$

- Unknowns: velocity $v(t,x) \in \mathbb{R}^d$, and pressure $p(t,x) \in \mathbb{R}$.
- Viscosity $\nu > 0$.
- Pressure and velocity: $p = -\sum_{i,j=1}^d \partial_i\partial_j\Delta^{-1}(v^i v^j)$.
- Scaling invariance: for $\lambda > 0$, $(\lambda v(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x))$.
- **Fundamental results:** Leray (Weak global solution if $v_0 \in L^2(\mathbb{R}^d)$, uniqueness when $d = 2$) and Fujita-Kato (Unique strong local solution if $v_0 \in \dot{H}^{\frac{d}{2}-1}$, global solution for small data).

Presentation of geophysical models

- Geophysical fluids: Rotation of the Earth, vertical stratification of the density.
- Scales, Rossby and Froude numbers.
- Small parameters $Ro = \varepsilon$, $Fr = \varepsilon F$ $(F > 0)$

Primitive system

- $U_\varepsilon(t,x) = (v_\varepsilon, \theta_\varepsilon) = (v_\varepsilon^1, v_\varepsilon^2, v_\varepsilon^3, \theta_\varepsilon)$,
- Velocity: $v_\varepsilon(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,
- Scalar potential temperature: $\theta_\varepsilon(t,x)$,
- Geopotential: $\phi_\varepsilon(t,x)$.

Primitive system

Primitive system

$$\begin{cases} \partial_t U_\varepsilon + U_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon}\mathcal{A}U_\varepsilon = \frac{1}{\varepsilon}(-\nabla\Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_{\varepsilon|t=0} = U_{0,\varepsilon}. \end{cases} \qquad (PE_\varepsilon)$$

where $U_\varepsilon \cdot \nabla U_\varepsilon \overset{def}{=} v_\varepsilon \cdot \nabla U_\varepsilon$ and

$$LU_\varepsilon \overset{\text{def}}{=} (\nu\Delta v_\varepsilon, \nu'\Delta\theta_\varepsilon), \quad \mathcal{A} \overset{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

Primitive system

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Rotating fluids system

Rotating fluids system

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon - \nu \Delta v_\varepsilon + \frac{e_3 \wedge v_\varepsilon}{\varepsilon} = -\frac{1}{\varepsilon} \nabla p_\varepsilon, \\ \mathrm{div}\, v_\varepsilon = 0, \\ v_{\varepsilon|t=0} = v_0. \end{cases} \qquad (RF_\varepsilon)$$

Remarks

- The terms $\mathcal{A}U_\varepsilon$ et $(\nabla\Phi_\varepsilon, 0)$ are said to be penalized and lead the asymptotics together with the divergence-free condition.
- $\mathcal{A}$ skewsymmetric, energy methods easily adapted to obtain Leray and Fujita-Kato results in the spaces ($s \in \mathbb{R}$, $T \in]0, \infty]$):

$$\begin{cases} \dot{E}_T^s = \mathcal{C}([0,T], \dot{H}^s) \cap L^2([0,T], \dot{H}^{s+1}), \\ \|f\|_{\dot{E}_T^s}^2 \overset{def}{=} \|f\|_{L_T^\infty \dot{H}^s}^2 + \min(\nu, \nu')\|f\|_{L_T^2 \dot{H}^{s+1}}^2. \end{cases}$$

Homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ endowed with the norm $\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$.

Study of the asymptotics when $\varepsilon \to 0$

Procedure

- The penalized terms impose a limit system and a special structure/decomposition linked with it,
- Notion of "well-/ill-prepared" initial data,
- Globally well-posed limit system (strong solutions),
- Better results for the lifespan of strong solutions are transmitted (for strong enough rotation/stratification i.-e. $\varepsilon \to 0$),
- Convergence rates.

First approach: well-prepared initial data

- J.-L. Lions, R. Temam, S. Wang ('92, '94),
- T. Beale, A. Bourgeois ('94),
- P. Embid, A. Majda ('96, '98),
- E. Grenier ('97)
- B. Desjardins, E. Grenier ('98),
- I. Gallagher ('98),
- A. Babin, A. Mahalov, B. Nicolaenko ('96, '99, '01).

Dispersive approach: ill-prepared initial data for (RF_ε)

- J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier ('00, '02, '02 (Ekman),'06),
- A. Dutrifoy ('05),
- V.-S. Ngo ($\nu \to 0$) ('09),
- M. Hieber, Y. Shibata ('10),
- T. Iwabuchi, R. Takada ('15, '13, '14),
- Y. Koh, S. Lee, R. Takada (Littman) ('14)
- FC ('23)

see also:

- I. Gallagher, L. Saint Raymond ('06, '06),
- I. Gallagher ('08)

Dispersive approach: ill-prepared initial data for (PE_ε)

- A. Dutrifoy ('04),
- FC ('05, '04, '06, '08, '16, '18, '18, '20, '23),
- FC, V.-S. Ngo ('11),
- H. Koba, A. Mahalov, T. Yoneda ($\nu = \nu'$, '12),
- T. Iwabuchi, A. Mahalov, R. Takada ($\nu = \nu'$, '17),
- S. Scrobogna ($\mathbb{T}^3$, '18),

Special case: F=1,

- J.-Y. Chemin ('97, $\nu \sim \nu'$),
- D. Iftimie (F=1, $\nu = \nu' = 0$) ('99)
- FC ('18, general case)

Asymptotics for the Rotating fluids

Limit system: 2D-NS with 3 components (Global strong solutions)

$$\begin{cases} \partial_t \bar{u}_h + \bar{u}_h \cdot \nabla_h \bar{u}_h - \nu \Delta_h \bar{u}_h = -\nabla_h \bar{p}, \\ \partial_t \bar{u}_3 + \bar{u}_h \cdot \nabla_h \bar{u}_3 - \nu \Delta_h \bar{u}_3 = 0, \\ \mathrm{div}\,_h \bar{u}_h = 0, \\ \bar{u}_{|t=0} = \bar{u}_0. \end{cases} \qquad (2D-NS)$$

Asymptotics (Chemin, Desjardins, Gallagher, Grenier, 2002)

- $v_{\varepsilon|t=0} = v_0(x) + \bar{u}_0(x_h)$.
- Direct study of $v_\varepsilon - \bar{u} - W_\varepsilon$, where W_ε solves

$$\begin{cases} \partial_t W_\varepsilon - \nu \Delta W_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}(e_3 \wedge W_\varepsilon) = 0, \\ W_{\varepsilon|t=0} = v_0. \end{cases}$$

Asymptotics for the Rotating fluids

- Taylor-Proudman (Physics) theorem which states for strong rotation a column structure (that is a limit velocity independant of x_3).
- We have to consider non-conventional initial data to reach such limit (for "Navier-Stokes"-classical initial data $v_0 \in L^2(\mathbb{R}^3)$ or $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, the limit is zero)

Asymptotics for the Primitive system

Limit system: Quasi-geostrophic system

$$\begin{cases} \partial_t \widetilde{\Omega}_{QG} + \widetilde{v}_{QG}.\nabla \widetilde{\Omega}_{QG} - \Gamma \widetilde{\Omega}_{QG} = 0, \\ \widetilde{U}_{QG} = (\widetilde{v}_{QG}, \widetilde{\theta}_{QG}) = (-\partial_2, \partial_1, 0, -F\partial_3)\Delta_F^{-1}\widetilde{\Omega}_{QG}, \end{cases} \qquad (QG)$$

Special structure: from the potential vorticity:

$$\Omega(U) \overset{def}{=} \partial_1 v^2 - \partial_2 v^1 - F\partial_3\theta,$$

we define the quasi-geostrophic and oscillating/oscillatory parts of a 4-components function U:

$$U_{QG} \overset{def}{=} \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1}\Omega(U), \quad \text{and } U_{osc} \overset{def}{=} U - U_{QG}. \qquad (1)$$

Asymptotics for the Primitive system

Asymptotics (FC '04 →' 23)

- Global strong solutions for the limit system (no stretching)
- $U_{\varepsilon|t=0} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc}$ (ill-prepared).
- Direct study of $U_\varepsilon - \widetilde{U}_{QG} - W_\varepsilon$, where W_ε solves

$$\begin{cases} \partial_t W_\varepsilon - LW_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}W_\varepsilon = -G^b - G^l, \\ W_{\varepsilon|t=0} = U_{0,\varepsilon,osc} \end{cases}$$

- Frequency truncation when $\nu \neq \nu'$.
- Weak, strong solutions, potential vorticity patches, convergence rates, anisotropic viscosities...

Strongly stratified Boussinesq model without rotation

Strongly stratified Boussinesq system

$$\begin{cases} \partial_t U_\varepsilon + U_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon}\mathcal{B}U_\varepsilon = \frac{1}{\varepsilon}(-\nabla\Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_{\varepsilon|t=0} = U_{0,\varepsilon}. \end{cases} \qquad (S_\varepsilon)$$

$$LU_\varepsilon \stackrel{\text{def}}{=} (\nu\Delta v_\varepsilon, \nu'\Delta\theta_\varepsilon), \quad \mathcal{B} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Weak and strong solutions

For any fixed $\varepsilon > 0$

Theorem (J. Leray, 1933)

If $U_{0,\varepsilon} \in L^2(\mathbb{R}^3)$, then there exists a Leray solution $U_\varepsilon \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ (+energy).

No uniqueness ($d = 3$).

Theorem (H. Fujita and T. Kato, 1963, scaling invariance)

If $U_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then there exists a unique maximal lifespan $T_\varepsilon^* > 0$ and a unique solution $U_\varepsilon \in \mathcal{C}_T \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap L_T^2 \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ for all $T < T_\varepsilon^*$.
+ blow-up criteria and weak-strong uniqueness.

Previous results, inviscid case $\nu = \nu' = 0$

- K. Widmayer (CMS 2018): if U_ε is a regular bounded solution, it converges to $(\bar{u}(x), 0, 0)$ where $\bar{u} : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^2$ solves ($\mathbb{P}_2$ orthogonal projector onto horizontal divergence free vectorfields):

$$\begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla_h \bar{u} = -\nabla_h \bar{p}, \\ \operatorname{div}_h \bar{u} = 0, \\ \bar{u}_{|t=0} = (\mathbb{P}_2 U_0)^h, \end{cases} \tag{2}$$

- R. Takada (ARMA 2019): Existence result and convergence rate:

$$\|U_\varepsilon - (\bar{u}, 0, 0)\|_{L^q_T W^{1,\infty}} \leq C\varepsilon^{\frac{1}{q}}.$$

- S. Lee, R. Takada (IUMJ 2017): $(\nu = \nu')$
  Let $s \in]\frac{1}{2}, \frac{5}{8}]$. There exists $\delta_1, \delta_2 > 0$ such that for any initial data U_0 such that $\mathbb{P}_2 U_0 \in \dot{H}^{\frac{1}{2}}$, $U_{0,osc} \overset{def}{=} (I_d - \mathbb{P}_2)U_0 \in \dot{H}^s$ satisfy:

$$\|\mathbb{P}_2 U_0\|_{\dot{H}^{\frac{1}{2}}} \leq \delta_2, \text{ and } \|U_{0,osc}\|_{\dot{H}^s} \leq \delta_1 \varepsilon^{-\frac{1}{2}(s-\frac{1}{2})},$$

  there exists a unique global **mild solution** $U_\varepsilon \in L^4(\dot{W}^{\frac{1}{2},3})$.

if $\|\mathbb{P}_2 U_0\|_{\dot{H}^{\frac{1}{2}}}$ is sufficiently small, there exists a global solution for small enough ε.

- S. Scrobogna (DCDS 2020): Let $U_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ with $U_{0,S} = \mathbb{P}_2 U_0 \in H^1(\mathbb{R}^3)$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, there exists a unique global solution $U_\varepsilon \in \dot{E}^{\frac{1}{2}}$. Moreover, U_ε converges to $(\widetilde{v}^h, 0, 0)$, where $\widetilde{v}^h$ is the unique global solution of the two-component Navier-Stokes system:

$$\begin{cases} \partial_t \widetilde{v}^h + \widetilde{v}^h \cdot \nabla_h \widetilde{v}^h - \nu \Delta \widetilde{v}^h & = -\nabla_h \widetilde{\pi}^0, \\ \operatorname{div}_h \widetilde{v}^h = 0, \\ \widetilde{v}^h_{|t=0} = \mathbb{P}_2 U_0, \end{cases} \tag{3}$$

Something surprizing...

Extension and question

Question: why does the previous limit not depend on ν' ?

Before answering this question, let us precisely see how is obtained the limit system.

Formal approach for the limit: rewriting the pressure

Taking the divergence of the velocity part of (S_ε) we can separate the geopotential into $\Phi_\varepsilon \overset{def}{=} P_\varepsilon^1 + \varepsilon P_\varepsilon^0$, where:

$$\begin{cases} P_\varepsilon^1 = -\Delta^{-1}\partial_3\theta_\varepsilon, \\ P_\varepsilon^0 = -\sum_{i,j=1}^3 \partial_i\partial_j\Delta^{-1}(v_\varepsilon^i v_\varepsilon^j), \end{cases} \tag{4}$$

leading to the following rewriting:

$$\begin{cases} \partial_t v_\varepsilon^1 + v_\varepsilon \cdot \nabla v_\varepsilon^1 - \nu\Delta v_\varepsilon^1 & = -\partial_1 P_\varepsilon^0 - \frac{1}{\varepsilon}\partial_1 P_\varepsilon^1, \\ \partial_t v_\varepsilon^2 + v_\varepsilon \cdot \nabla v_\varepsilon^2 - \nu\Delta v_\varepsilon^2 & = -\partial_2 P_\varepsilon^0 - \frac{1}{\varepsilon}\partial_2 P_\varepsilon^1, \\ \partial_t v_\varepsilon^3 + v_\varepsilon \cdot \nabla v_\varepsilon^3 - \nu\Delta v_\varepsilon^3 & = -\partial_3 P_\varepsilon^0 - \frac{1}{\varepsilon}(\partial_3 P_\varepsilon^1 + \theta_\varepsilon), \\ \partial_t \theta_\varepsilon + v_\varepsilon \cdot \nabla \theta_\varepsilon - \nu'\Delta \theta_\varepsilon & = \frac{1}{\varepsilon} v_\varepsilon^3, \\ \operatorname{div} v_\varepsilon & = 0. \end{cases} \tag{5}$$

Formal approach for the limit: dealing with the penalized terms

Assuming that $(v_\varepsilon, \theta_\varepsilon, P_\varepsilon^0, P_\varepsilon^1) \underset{\varepsilon\to 0}{\longrightarrow} (\widetilde{v}, \widetilde{\theta}, \widetilde{P}^0, \widetilde{P}^1)$ in a sufficiently strong way (for derivatives and nonlinear terms...) if we hope in addition that:

$$\begin{cases} -\frac{1}{\varepsilon}\partial_1 P_\varepsilon^1 & \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{X}, \\ -\frac{1}{\varepsilon}\partial_2 P_\varepsilon^1 & \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{Y}, \end{cases} \quad \begin{cases} -\frac{1}{\varepsilon}(\partial_3 P_\varepsilon^1 + \theta_\varepsilon) & \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{Z}, \\ \frac{1}{\varepsilon} v_\varepsilon^3 & \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{T}, \end{cases} \tag{6}$$

we need that:

$$\begin{cases} \partial_1 \widetilde{P}^1 = \partial_2 \widetilde{P}^1 = 0, \\ \partial_3 \widetilde{P}^1 + \widetilde{\theta} = 0, \\ \widetilde{v}^3 = 0, \end{cases} \tag{7}$$

Formal approach for the limit: dealing with the penalized terms

which implies:

$$\begin{cases} \widetilde{P}^1 \text{ and } \widetilde{\theta} = -\partial_3 \widetilde{P}^1 \text{ only depend on } x_3, \\ \widetilde{v}^3 = 0. \end{cases} \tag{8}$$

Additionnally, $\widetilde{P}^0 = -\sum_{i,j=1}^{2} \Delta^{-1}\partial_i\partial_j(\widetilde{v}^i \widetilde{v}^j)$ and defining $\widetilde{v}^h \overset{def}{=} (\widetilde{v}^1, \widetilde{v}^2)$, we have:

$$\mathrm{div}_h \widetilde{v}^h \overset{def}{=} \partial_1 \widetilde{v}^1 + \partial_2 \widetilde{v}^2 = 0.$$

Formal approach for the limit: how to obtain the limit system ?

The limit system turns into:

$$\begin{cases} \partial_t \widetilde{v}^1 + \widetilde{v}^h \cdot \nabla_h \widetilde{v}^1 - \nu \Delta \widetilde{v}^1 & = -\partial_1 \widetilde{P}^0 + \widetilde{X}, \\ \partial_t \widetilde{v}^2 + \widetilde{v}^h \cdot \nabla_h \widetilde{v}^2 - \nu \Delta \widetilde{v}^2 & = -\partial_2 \widetilde{P}^0 + \widetilde{Y}, \\ 0 & = -\partial_3 \widetilde{P}^0 + \widetilde{Z}, \\ \partial_t \widetilde{\theta} - \nu' \partial_3^2 \widetilde{\theta} & = \widetilde{T}, \\ \mathrm{div}\,_h \widetilde{v} & = 0. \end{cases} \tag{9}$$

Formal approach for the limit: How to get rid of parameters ?

Using once more the divergence-free (and 2d-divergence-free) conditions and the vorticity (see later), we obtain that:

$$\begin{cases} \partial_1 \widetilde{X} + \partial_2 \widetilde{Y} + \partial_3 \widetilde{Z} = 0. \\ \partial_1 \widetilde{Y} - \partial_2 \widetilde{X} = 0, \end{cases}$$

wich formally leads to (we recall that $\widetilde{Z} = \partial_3 \widetilde{P}^0$):

$$(\widetilde{X}, \widetilde{Y}) = -\nabla_h \partial_3^2 \Delta_h^{-1} \widetilde{P}^0.$$

Writing the limit system

Limit system

Gathering the previous informations, the formal limit is written as:

$$\begin{cases} \partial_t \widetilde{v}^h + \widetilde{v}^h \cdot \nabla_h \widetilde{v}^h - \nu \Delta \widetilde{v}^h & = -\nabla_h \widetilde{\pi}^0, \\ \mathrm{div}\,_h \widetilde{v}^h = 0, \end{cases}$$

$$\partial_t \widetilde{\theta} - \nu' \partial_3^2 \widetilde{\theta} = \widetilde{T},$$

where $\widetilde{\pi}^0 = \Delta_h^{-1} \Delta \widetilde{P}^0 = -\sum_{i,j=1}^2 \Delta_h^{-1} \partial_i \partial_j (\widetilde{v}^i \widetilde{v}^j)$ and $\widetilde{P}^1, \widetilde{\theta}, \widetilde{T}$ only depend on (t, x_3).

Limit system

Remarks

- Nothing forces $\widetilde{\theta}(x_3)$ to be zero.
- How to deal with $\widetilde{T} = \lim_{\varepsilon \to 0} \frac{v_\varepsilon^3}{\varepsilon}$? Treat it like a parameter.
- We will consider the case $\widetilde{T} = 0$ and initial data according to:

$$U_{\varepsilon|t=0}(x) = U_{0,\varepsilon}(x) + (0,0,0,\widetilde{\theta}_{0,\varepsilon}(x_3)).$$

- Vorticity formulation: if $\widetilde{\omega} = \omega(\widetilde{v}) = \partial_1 \widetilde{v}^2 - \partial_2 \widetilde{v}^1$ we rewrite the velocity system as follows:

$$\begin{cases} \partial_t \widetilde{\omega} + \widetilde{v}^h \cdot \nabla_h \widetilde{\omega} - \nu \Delta \widetilde{\omega} = 0, \\ \widetilde{v}^h = \nabla_h^\perp \Delta_h^{-1} \widetilde{\omega}. \end{cases}$$

The vorticity formulation suggests the following structure:

Stratified/Oscillating decomposition

If f is a R^4-valued function, its vorticity is defined by:

$$\omega(f) = \partial_1 f^2 - \partial_2 f^1.$$

From this we define (denoting $\mathrm{div}_h f^h \stackrel{def}{=} \partial_1 f^1 + \partial_2 f^2$):

$$f_S = \mathbb{P}_2 f = \begin{pmatrix} \nabla_h^\perp \Delta_h^{-1} \omega(f) \\ 0 \\ 0 \end{pmatrix}, \text{ and}$$

$$f_{osc} = f - f_S = (I_d - \mathbb{P}_2) f = \begin{pmatrix} \nabla_h \Delta_h^{-1} \mathrm{div}_h f^h \\ f^3 \\ f^4 \end{pmatrix}.$$

Statement of the results: aim of our study

Aim: Prove that for the following initial data

$$U_{\varepsilon|t=0}(x) = U_{0,\varepsilon,S}(x) + U_{0,\varepsilon,osc}(x) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \widetilde{\theta}_{0,\varepsilon}(x_3) \end{pmatrix},$$

with:

$$\begin{cases} U_{0,\varepsilon,S}(x) \underset{\varepsilon\to 0}{\longrightarrow} (\widetilde{v}_0^h(x), 0, 0), \\ \widetilde{\theta}_{0,\varepsilon}(x_3) \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{\theta}_0(x_3), \end{cases} \qquad (f^h = (f^1, f^2))$$

the solutions become global and converge (as $\varepsilon \to 0$) towards those of the following system:

Aim of our study

$$\begin{cases} \partial_t \widetilde{v}^h + \widetilde{v}^h \cdot \nabla_h \widetilde{v}^h - \nu \Delta \widetilde{v}^h & = -\nabla_h \widetilde{\pi}^0, \\ \mathrm{div}\,_h \widetilde{v}^h = 0, \\ \widetilde{v}^h_{|t=0} = \widetilde{v}^h_0, \end{cases} \tag{10}$$

and

$$\begin{cases} \partial_t \widetilde{\theta} - \nu' \partial_3^2 \widetilde{\theta} = 0, \\ \widetilde{\theta}_{|t=0} = \widetilde{\theta}_0. \end{cases} \tag{11}$$

Remarks:

- (11) is globally well-posed when $\widetilde{\theta}_0 \in \dot{B}^s_{2,1}(\mathbb{R})$ (for any $s \in \mathbb{R}$).
- (10) is globally well-posed when $\widetilde{v}^h_0 \in H^{\frac{1}{2}+\delta}$ ($\mathbb{R}^2$-valued) with $\mathrm{div}\,_h \widetilde{v}^h_0 = 0$ (for $\delta > 0$).

What system to study ?

To simplify, we assume in this talk that $\widetilde{\theta}_{0,\varepsilon}(x_3) = \widetilde{\theta}_0(x_3)$ and $U_{0,\varepsilon,S}(x) = (\widetilde{v}_0^h, 0, 0)$ so that:

$$U_{\varepsilon|t=0}(x) = U_{0,\varepsilon}(x) + (0, 0, 0, \widetilde{\theta}_0(x_3)),$$

where

$$U_{0,\varepsilon}(x) = (\widetilde{v}^h(x), 0, 0) + U_{0,\varepsilon,osc}(x).$$

Problem: the classical theorems are not able to deal with unconventional initial data:

$$U_{0,\varepsilon}(x_1, x_2, x_3) + (0, 0, 0, \widetilde{\theta}_0(x_3)),$$

Rewriting the limit system

Setting $\widetilde{U} \stackrel{def}{=} (\widetilde{v}^h, 0, \widetilde{\theta})$,

Final form of the limit system:

$$\begin{cases} \partial_t \widetilde{U} + \widetilde{U} \cdot \nabla \widetilde{U} - L\widetilde{U} + \frac{1}{\varepsilon}\mathcal{B}\widetilde{U} = -\widetilde{G} - \begin{pmatrix} \nabla \widetilde{g} \\ 0 \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} \nabla \widetilde{P}^1 \\ 0 \end{pmatrix}, \\ \operatorname{div} \widetilde{v} = 0, \\ \widetilde{U}_{|t=0} = (\widetilde{v}_0^h, 0, \widetilde{\theta}_0). \end{cases}$$

where

$$\widetilde{G} = \mathbb{P} \begin{pmatrix} \partial_1 \widetilde{\pi}^0 \\ \partial_2 \widetilde{\pi}^0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1 \partial_3^2 \Delta^{-1} \Delta_h^{-1} \widetilde{q}_0 \\ \partial_2 \partial_3^2 \Delta^{-1} \Delta_h^{-1} \widetilde{q}_0 \\ -\partial_3 \Delta^{-1} \widetilde{q}_0 \\ 0 \end{pmatrix} \sim \widetilde{v}^h \cdot \nabla \widetilde{v}^h.$$

What system to study ?

Putting $D_\varepsilon \overset{def}{=} U_\varepsilon - \widetilde{U}$ and $V_\varepsilon = (D_\varepsilon^1, D_\varepsilon^2, D_\varepsilon^3)$, we will study:

$$\begin{cases} \partial_t D_\varepsilon - LD_\varepsilon + \frac{1}{\varepsilon}\mathcal{B}D_\varepsilon = \widetilde{G} - \begin{pmatrix} \nabla q_\varepsilon \\ 0 \end{pmatrix} \\ \qquad - \left[D_\varepsilon \cdot \nabla D_\varepsilon + \begin{pmatrix} D_\varepsilon \cdot \nabla \widetilde{v}^h \\ 0 \\ D_\varepsilon^3 \cdot \partial_3 \widetilde{\theta} \end{pmatrix} + \widetilde{v}^h \cdot \nabla_h D_\varepsilon \right] \\ \mathrm{div}\, V_\varepsilon = 0, \\ D_{\varepsilon|t=0} = U_{0,\varepsilon,osc}. \end{cases} \tag{12}$$

Classical initial data.

Study of the limit systems

Theorem (1D-Heat equation):

Let $s \in \mathbb{R}$. For any $\widetilde{\theta}_0 \in \dot{B}^s_{2,1}(\mathbb{R})$ there exists a unique global solution $\widetilde{\theta}$ of (11) and for all $t \geq 0$, we have:

$$\|\widetilde{\theta}\|_{\widetilde{L}^\infty_t \dot{B}^s_{2,1}} + \nu' \|\widetilde{\theta}\|_{L^1_t \dot{B}^{s+2}_{2,1}} \leq \|\widetilde{\theta}_0\|_{\dot{B}^s_{2,1}}. \tag{13}$$

More generally for $s \in \mathbb{R}$ and $p, r \in [1, \infty]$, there exists a constant $C > 0$ such that if $\widetilde{\theta}_0 \in \dot{B}^s_{p,r}(\mathbb{R})$ then for all $q \in [1, \infty]$

$$\|\widetilde{\theta}\|_{\widetilde{L}^q_t \dot{B}^{s+\frac{2}{q}}_{p,r}} \leq \frac{C}{(\nu')^{\frac{1}{q}}} \|\widetilde{\theta}_0\|_{\dot{B}^s_{p,r}}. \tag{14}$$

Study of the limit systems

Theorem (Velocity system, FC '23):

Let $\delta > 0$ and $\widetilde{v}_0^h \in H^{\frac{1}{2}+\delta}$ ($\mathbb{R}^2$-valued) with $\operatorname{div}_h \widetilde{v}_0^h = 0$. System (10) has a unique global solution $\widetilde{v}^h \in E^{\frac{1}{2}+\delta} = \dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$ and $\exists C = C_{\delta,\nu} > 0, t \geq 0$:

$$\|\widetilde{v}^h\|^2_{L^\infty H^{\frac{1}{2}+\delta}} + \nu \|\nabla \widetilde{v}^h\|^2_{L^2 H^{\frac{1}{2}+\delta}} \leq C_{\delta,\nu} \|\widetilde{v}_0^h\|^2_{H^{\frac{1}{2}+\delta}} \max(1, \|\widetilde{v}_0^h\|^{\frac{1}{\delta}}_{H^{\frac{1}{2}+\delta}})$$
$$\leq C_{\delta,\nu} \max(1, \|\widetilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}})^{2+\frac{1}{\delta}}, \quad (15)$$

Moreover, we can also bound the term $\widetilde{G}$: for all $s \in [0, \frac{1}{2}+\delta]$,

$$\int_0^\infty \|\widetilde{G}(\tau)\|_{\dot{H}^s} d\tau \leq C_{\delta,\nu} \max(1, \|\widetilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}})^{2+\frac{1}{\delta}}. \quad (16)$$

Existence of local strong (Fujita-Kato) solutions

Theorem (Existence of local strong solutions, FC CDPE '24)

Let $\varepsilon > 0$, $\delta \in]0,1]$, $\widetilde{v}_0^h \in H^{\frac{1}{2}+\delta}(\mathbb{R}^3)$ and for some fixed $\beta > 0$, $\widetilde{\theta}_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R}) \cap \dot{B}_{2,1}^{-\frac{1}{2}+\beta}(\mathbb{R})$. For any $U_{0,\varepsilon} = U_{0,\varepsilon,S} + U_{0,\varepsilon,osc} \in H^{\frac{1}{2}}$, there exists a unique local solution D_ε of (12) with lifespan $T_\varepsilon^* > 0$ such that for any $T < T_\varepsilon^*$, $D_\varepsilon \in E_T^{\frac{1}{2}} = \dot{E}_T^0 \cap \dot{E}_T^{\frac{1}{2}}$. Moreover, the following properties are true:

- **Regularity propagation:** if in addition $U_{0,\varepsilon} \in \dot{H}^s$ for some $s \in [0, \frac{1}{2}+\delta]$ then for any $T < T_\varepsilon^*$, $D_\varepsilon \in \dot{E}_T^0 \cap \dot{E}_T^s$.
- **Blow-up criterion:** $\int_0^{T_\varepsilon^*} \|\nabla D_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau < \infty \Longrightarrow T_\varepsilon^* = \infty$.

And now for the initial data $U_{\varepsilon|t=0} = (\widetilde{v}_0(x), 0, \widetilde{\theta}_0(x_3)) + U_{0,\varepsilon,osc}$,

Global existence and convergence: simplified statement

Theorem (Global existence and convergence, FC CDPE '24)

For all $\nu, \nu', \mathbb{C}_0 > 0$, $\delta \in]0, \frac{1}{8}]$, $\widetilde{v}_0^h$, $\widetilde{\theta}_0$ and $U_{0,\varepsilon,osc}$ with,

$$\|\widetilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \leq \mathbb{C}_0 \quad \text{and} \quad \|\widetilde{\theta}_0\|_{\dot{B}_{2,1}^{-\frac{3}{4}}(\mathbb{R})\cap \dot{B}_{2,1}^{-\frac{1}{4}+\delta}(\mathbb{R})} \leq \mathbb{C}_0,$$

there exist $\varepsilon_0, K, \gamma, c, \mathbb{D}_0, q > 0$ such that if (for any $\varepsilon > 0$)

$$\|U_{0,\varepsilon,osc}\|_{L^q} + \||D|^{\frac{1}{2}} U_{0,\varepsilon,osc}\|_{L^q} + \|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta}\cap \dot{H}^{\frac{1}{2}+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma}, \tag{17}$$

then for any $\varepsilon \in]0, \varepsilon_0]$, there exists a unique global strong solution U_ε of (S_ε) which satisfies $U_\varepsilon - (\widetilde{v}^h, 0, \widetilde{\theta}) \in \dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\frac{\delta}{2}}$ and

$$\|U_\varepsilon - (\widetilde{v}^h, 0, \widetilde{\theta})\|_{L^2(\mathbb{R}_+, L^\infty(\mathbb{R}^3))} \leq \mathbb{D}_0 \varepsilon^K.$$

Link with the classical Boussinesq system

Our system is related to:

The classical Boussinesq system

$$\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v + \kappa^2 \rho e_3 = -\nabla P, \\ \partial_t \rho + v\cdot\nabla\rho - \nu'\Delta\rho = 0, \\ \text{div } v = 0. \end{cases} \tag{18}$$

Explicit stationary solution: $(\bar{V}_\varepsilon, \bar{P}_\varepsilon)$ with
$\bar{P}_\varepsilon(x_3) = \bar{P}_{0,\varepsilon} - \kappa^2 \bar{\rho}_{0,\varepsilon} x_3 + \frac{x_3^2}{2\varepsilon^2}$,

$$\bar{V}_\varepsilon(x_3) = \begin{pmatrix} 0 \\ \bar{\rho}_\varepsilon(x_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\rho}_{0,\varepsilon} - \frac{x_3}{\varepsilon^2\kappa^2} \end{pmatrix},$$

Change of variable: solutions near $(\bar{V}_\varepsilon, \bar{P}_\varepsilon)$:
$(V_\varepsilon, P_\varepsilon) = (v_\varepsilon, \rho_\varepsilon, P_\varepsilon)$ solves (18) if, and only if,
$(U_\varepsilon, \Phi_\varepsilon) = (v_\varepsilon, \theta_\varepsilon, \Phi_\varepsilon)$ solves (S_ε), where:

$$\theta_\varepsilon(x) \stackrel{def}{=} \varepsilon\kappa^2(\rho_\varepsilon(x) - \bar{\rho}_\varepsilon(x_3)), \quad \Phi_\varepsilon(x) = \varepsilon(P_\varepsilon(x) - \bar{P}_\varepsilon(x_3)).$$

$$\left[\text{that is } V_\varepsilon(x) = \begin{pmatrix} v_\varepsilon(x) \\ \rho_\varepsilon(x) \end{pmatrix} = \begin{pmatrix} 0 + v_\varepsilon(x) \\ \bar{\rho}_\varepsilon(x_3) + \frac{\theta_\varepsilon(x)}{\varepsilon\kappa^2} \end{pmatrix}\right],$$

Put differently, aside from its own geophysical interest, studying (S_ε) provides solutions for the Boussinesq system (18) near the explicit vertically stratified solution $(\bar{V}_\varepsilon, \bar{P}_\varepsilon)$.

The previous theorems can be rewritten as asymptotics results for the classical Boussinesq system as follows:

Theorem: Global strong solutions for Boussinesq

With the previous assumptions, for any $\varepsilon \in]0, \varepsilon_0]$, there exists a unique global strong solution $V_\varepsilon = (v_\varepsilon, \rho_\varepsilon)$ to the Boussinesq system corresponding to the following initial data:

$$V_\varepsilon|_{t=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \rho_{0,\varepsilon} - \frac{x_3}{\varepsilon^2 \kappa^2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\widetilde{\theta}_0(x_3)}{\varepsilon \kappa^2} \end{pmatrix} + \begin{pmatrix} \widetilde{v}_0^h(x) + v_{0,osc,\varepsilon}^h(x) \\ v_{0,osc,\varepsilon}^3(x) \\ \frac{\theta_{0,osc,\varepsilon}(x)}{\varepsilon \kappa^2} \end{pmatrix}.$$

Moreover, we have an **asymptotic expansion** of the solution $V_\varepsilon = (v_\varepsilon, \rho_\varepsilon)$ when ε goes to zero: if $U_{0,\varepsilon,osc}$ is of size $\varepsilon^{-\gamma}$ there exist some $K>0$ and a four-component function D_ε such that,

$$\|D_\varepsilon\|_{L^2(\mathbb{R}_+,L^\infty(\mathbb{R}^3))} \leq \mathbb{D}_0 \varepsilon^K, \text{ and}$$

$$V_\varepsilon(t,x) = \begin{pmatrix} D_\varepsilon^h(t,x) + \widetilde{v}^h(t,x) \\ D_\varepsilon^3(t,x) \\ \bar{\rho}_\varepsilon(x_3) + \frac{\widetilde{\theta}(t,x_3) + D_\varepsilon^4(t,x)}{\varepsilon\kappa^2} \end{pmatrix}.$$

This means that:

$$V_\varepsilon(t,x) \underset{\varepsilon\to 0}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\rho}_{0,\varepsilon} - \frac{x_3}{\varepsilon^2\kappa^2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\widetilde{\theta}(t,x_3)}{\varepsilon\kappa^2} \end{pmatrix} + \begin{pmatrix} \widetilde{v}^h(t,x) + \mathcal{O}(\varepsilon^K) \\ \mathcal{O}(\varepsilon^K) \\ \mathcal{O}(\varepsilon^{K-1}) \end{pmatrix}.$$

Precise statement in the case $\nu = \nu'$

Stating precisely the previous result requires that we introduce the following waves (As $\nu = \nu'$, $L = \nu\Delta$.):

$$\begin{cases} \partial_t W_\varepsilon - \nu \Delta W_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{B} W_\varepsilon = \widetilde{G}, \\ W_{\varepsilon|t=0} = U_{0,\varepsilon,osc}, \end{cases} \tag{19}$$

which

- take advantage of dispersion,
- allow to "eat" the constant external force term $\widetilde{G}$ (by making it oscillate/disperse at infinity).

Precise statement in the case $\nu = \nu'$

Global existence and convergence when $\nu = \nu'$, FC CPDE '24

For all $\nu, \mathbb{C}_0 > 0$, $\delta \in]0, \frac{1}{8}]$, $\widetilde{v}_0^h$, $\widetilde{\theta}_0$ and $U_{0,\varepsilon,osc}$ with,

$$\|\widetilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \leq \mathbb{C}_0 \quad \text{and} \quad \|\widetilde{\theta}_0\|_{\dot{B}_{2,1}^{-\frac{3}{4}}(\mathbb{R}) \cap \dot{B}_{2,1}^{-\frac{1}{4}+\delta}(\mathbb{R})} \leq \mathbb{C}_0,$$

then:

1- There exist $m_0, \varepsilon_0 > 0$ such that if for some $c > 0$ (as small as we want)

$$\|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq m_0 \varepsilon^{-\frac{\delta}{2}},$$

then for any $\varepsilon \in]0, \varepsilon_0]$, there exists a global solution of (S_ε) and $D_\varepsilon \in \dot{E}^0 \cap \dot{E}^{\frac{1}{2}}$.

Precise statement in the case $\nu = \nu'$

2- If there exists a function $m(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$ such that for some $c > 0$

$$\|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq m(\varepsilon)\varepsilon^{-\frac{\delta}{2}},$$

then if we define $\delta_\varepsilon = D_\varepsilon - W_\varepsilon$, there exists $\mathbb{D}_0 = \mathbb{D}_0(\nu, \mathbb{C}_0, \delta) > 0$ such that:

$$\|\delta_\varepsilon\|_{\dot{E}^0 \cap \dot{E}^{\frac{1}{2}}} \leq \mathbb{D}_0 \max\left(\varepsilon^{\frac{\delta}{2}}, m(\varepsilon)\right) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Precise statement in the case $\nu = \nu'$

3- Finally, if for some $c > 0$ and $\gamma \in]0, \frac{\delta}{2}[$ we have

$$\|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma},$$

then

$$\|\delta_\varepsilon\|_{\dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\frac{\delta}{2}-\gamma}} \leq \mathbb{D}_0 \varepsilon^{\frac{\delta}{2}-\gamma},$$

and for any $k \in]0,1[$ (as close to 1 as we wish), there exists $\mathbb{D}_0 = \mathbb{D}_0(\nu, \mathbb{C}_0, \delta, k) > 0$ such that:

$$\|D_\varepsilon\|_{L^2 L^\infty} = \|U_\varepsilon - (\widetilde{v}^h, 0, \widetilde{\theta})\|_{L^2 L^\infty} \leq \mathbb{D}_0 \varepsilon^{k(\frac{\delta}{2}-\gamma)}.$$

Ideas of the proof:

Everything relies on a bootstrap argument on the quantity $\delta_\varepsilon = D_\varepsilon - W_\varepsilon$, which satisfies:

$$\begin{cases} \partial_t \delta_\varepsilon - \nu \Delta \delta_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{B}\delta_\varepsilon = \displaystyle\sum_{i=1}^{10} G_i, \\ \delta_{\varepsilon|t=0} = 0, \end{cases} \tag{20}$$

where:

$$\begin{cases} G_1 \overset{def}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla \delta_\varepsilon), \quad G_2 \overset{def}{=} -\mathbb{P}((\delta_\varepsilon \cdot \nabla \widetilde{v}^h, 0, 0)), \\ G_3 \overset{def}{=} -\mathbb{P}(\widetilde{v}^h \cdot \nabla_h \delta_\varepsilon), \quad G_4 \overset{def}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla W_\varepsilon), \\ G_5 \overset{def}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla \delta_\varepsilon), \quad G_6 \overset{def}{=} -\mathbb{P}(\widetilde{v}^h \cdot \nabla_h W_\varepsilon), \\ G_7 \overset{def}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla \widetilde{v}^h, 0, 0), \quad G_8 \overset{def}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla W_\varepsilon), \\ G_9 \overset{def}{=} -\mathbb{P}(0, 0, 0, \delta_\varepsilon^3 \cdot \partial_3 \widetilde{\theta}), \quad G_{10} \overset{def}{=} -\mathbb{P}(0, 0, 0, W_\varepsilon^3 \cdot \partial_3 \widetilde{\theta}). \end{cases} \tag{21}$$

Ideas of the proofs: bootstrap

- Define

$$T_{\varepsilon,2} \stackrel{def}{=} \sup\left\{t \in [0, T_\varepsilon^*[/\ \forall t' \in [0,t],\ \|\delta_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{\nu_0}{4C}.\right\}$$

- Products of the form $a(x) \times \widetilde{\theta}(x_3)$ have to be dealt carefully.
- W_ε is small when viewed through special norms thanks to dispersion/Strichartz estimates ($L_t^p L_x^r$ or $\widetilde{L}_t^p \dot{B}_{r,q}^0$ for $1 \leq p \leq 8/(1-2/r)$).
- In the external force terms, we try to make these norms appear everywhere (never using energy estimates for W_ε) and we obtain:

Ideas of the proof:

A priori estimates, FC CPDE '24

$$\|\delta_\varepsilon(t)\|_{H^s}^2 + \frac{\nu}{2}\int_0^t \|\nabla\delta_\varepsilon(t')\|_{H^s}^2 dt'$$
$$\leq \mathbb{D}_0 \left(\|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^4 L^6}^2 + \|W_\varepsilon\|_{L_t^2 L^8}^2 + \|W_\varepsilon\|_{L_t^8 L_{v,h}^{\infty,2}}^2 \right)$$
$$\times \exp\left\{ \mathbb{D}_0 \left(1 + \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^4 L^6}^4 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^{\frac{2}{1-s}} + \|\nabla W_\varepsilon\|_{L_t^2 L^{\frac{8}{3}}}^2 \right) \right\} \quad (22)$$

- Here $s = \frac{1}{2}$ or $s = \frac{1}{2} + \eta\delta$.
- Note that we consider H^s (not the usual $\dot{H}^s$).

We show W_ε is small thanks to Strichartz estimates: there exists a constant $\mathbb{D}_0 = \mathbb{D}_0(\nu, \mathbb{C}_0, \delta, \eta) > 0$ such that for any $t \geq 0$,

$$\begin{cases} \|\nabla W_\varepsilon\|_{L_t^2 L^3} + \|W_\varepsilon\|_{L_t^4 L^6} \leq \mathbb{D}_0 \varepsilon^{\frac{\delta}{2}} \Big(\|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} + 1\Big), \\ \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6} \leq \mathbb{D}_0 \varepsilon^{(1-\eta)\frac{\delta}{2}} \Big(\|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} + 1\Big), \end{cases}$$

and

$$\varepsilon^{-\frac{1}{8}} \|W_\varepsilon\|_{L_t^2 L^8} + \|\nabla W_\varepsilon\|_{L_t^2 L^{\frac{8}{3}}} + \varepsilon^{-\frac{1}{16}} \|W_\varepsilon\|_{L_t^8 L_{v,h}^{\infty,2}} \leq \mathbb{D}_0 \varepsilon^{\frac{1}{16}} \Big(\|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}-c\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} + 1\Big). \quad (23)$$

End of the proof

- Injecting the previous estimates in the a priori estimates allows to close the bootstrap and prove that $T_{\varepsilon,2} = T_\varepsilon^* = \infty$.
- Convergence rates: $U_\varepsilon - (\widetilde{v}^h, 0, \widetilde{\theta}) = \delta_\varepsilon + W_\varepsilon$, and from the energy estimates, δ_ε is bounded in $L_t^2 L^\infty$ through

$$\dot{H}^{\frac{3}{2}-\alpha} \cap \dot{H}^{\frac{3}{2}+\beta} \hookrightarrow \dot{B}_{2,1}^{\frac{3}{2}} \hookrightarrow \dot{B}_{\infty,1}^0 \hookrightarrow L^\infty,$$

- We use once more the Strichartz estimates to bound W_ε in $L_t^2 \dot{B}_{\infty,1}^0$.

Strichartz estimates when $\nu = \nu'$

Consider:

$$\begin{cases} \partial_t f - (\nu\Delta - \frac{1}{\varepsilon}\mathbb{P}\mathcal{B})f = F_{ext}, \\ f_{|t=0} = f_0. \end{cases} \tag{24}$$

If $\nu = \nu'$, for all $\varepsilon > 0$, and all $\xi \in \mathbb{R}^3$, the matrix $\mathbb{B}(\xi, \varepsilon) = \widehat{\nu\Delta - \frac{1}{\varepsilon}}\mathbb{P}\mathcal{B}$ is diagonalizable and its eigenvalues satisfy:

$$\begin{cases} \lambda_1(\varepsilon, \xi) = 0, \\ \lambda_2(\varepsilon, \xi) = -\nu|\xi|^2, \\ \lambda_3(\varepsilon, \xi) = -\nu|\xi|^2 + i\frac{|\xi_h|}{\varepsilon|\xi|}, \\ \lambda_4(\varepsilon, \xi) = \overline{\lambda_3(\varepsilon, \xi)}, \end{cases} \tag{25}$$

The first eigenvector does not play any role, the rest are mutually orthogonal.

Strichartz estimates when $\nu = \nu'$

Isotropic Strichartz estimates

For any $d \in \mathbb{R}$, $r \geq 2$, $q \geq 1$, $\theta \in [0,1]$ and $p \in [1, \frac{4}{\theta(1-\frac{2}{r})}]$, there exists a constant $C = C_{p,r,\theta}$ such that for any f solving (24) for initial data f_0 and external force F_{ext} both with zero divergence and vorticity (that is in the kernel of $\mathbb{P}_2$), then

$$\||D|^d f\|_{\widetilde{L}_t^p \dot{B}_{r,q}^0} \leq \frac{C_{p,r,\theta}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{2}{r})}} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{r})} \left(\|f_0\|_{\dot{B}_{2,q}^{\sigma_1}} + \|F_{ext}\|_{\widetilde{L}_t^1 \dot{B}_{2,q}^{\sigma_1}} \right),$$

where $\sigma_1 = d + \frac{3}{2} - \frac{3}{r} - \frac{2}{p} + \frac{\theta}{2}(1 - \frac{2}{r})$.

Strichartz estimates when $\nu = \nu'$

Anisotropic Strichartz estimates

For any $d \in \mathbb{R}$, $m > 2$, $\theta \in [0,1]$ and $p \in [1, \frac{8}{\theta(1-\frac{2}{m})}]$, there exists a constant $C_{p,m,\theta}$ such that for any f solving (24) for initial data f_0 and external force F_{ext} such that $\mathrm{div}\, f_0 = \mathrm{div}\, F_{ext} = 0$ and $\omega(f_0) = \omega(F_{ext}) = 0$, then

$$\||D|^d f\|_{L_t^p L_{v,h}^{m,2}} \leq \frac{C_{p,m,\theta}}{\nu^{\frac{1}{p}-\frac{\theta}{8}(1-\frac{2}{m})}} \varepsilon^{\frac{\theta}{8}(1-\frac{2}{m})} \left(\|f_0\|_{\dot{B}^{\sigma_2}_{2,q}} + \|F_{ext}\|_{\widetilde{L}_t^1 \dot{B}^{\sigma_2}_{2,q}}\right), \tag{26}$$

where $\sigma_2 = d + \frac{1}{2} - \frac{1}{m} - \frac{2}{p} + \frac{\theta}{4}(1 - \frac{2}{m})$.

Thank you for your attention !

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For $0 < \alpha < R$, and $\beta \geq 0$, let us define, for any $x \in \mathbb{R}$,

$$f_\alpha(x) = \frac{\alpha x}{(x^2 + \alpha^2)^{\frac{3}{2}}},$$

and

$$I_{\alpha,\beta}^R(\sigma) \overset{def}{=} \int_0^{\sqrt{R^2-\alpha^2}} \frac{dx}{1 + \sigma(f_\alpha(x) - \beta)^2}, \tag{27}$$

Proposition (FC, 2023)

There exists a constant $C_0 > 0$ such that for any $\alpha > 0$, $R \geq \frac{2}{\sqrt{3}}\alpha$,

$$\sup_{\beta \in \mathbb{R}_+} I^R_{\alpha,\beta}(\sigma) \leq C_0 \frac{R^7}{\alpha^{\frac{11}{2}}} \min(1, \sigma^{-\frac{1}{4}}). \tag{28}$$

Moreover, the exponent $-\frac{1}{4}$ is optimal in the sense that there exist $c_0, \sigma_0 > 0$ such that for any $R \geq \frac{\sqrt{3}}{\sqrt{2}}\alpha$ and $\sigma \geq \sigma_0$,

$$\sup_{\beta \in \mathbb{R}_+} I^R_{\alpha,\beta}(\sigma) \geq c_0 \sigma^{-\frac{1}{4}} \alpha^{\frac{3}{2}}.$$