Some nonlinear heat equations with exponential non-linearity and with singular data in two dimensions

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The problem:

Consider the Cauchy problem for a semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } (0,\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{1}$$

where

- $u(t,x) : (0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ unknown function
- $f : \mathbb{R} \to \mathbb{R}$ continuous: nonlinearity
- u_0 given initial data.

Bounded initial data:

Assume

- initial data $u_0 \in L^\infty(\mathbb{R}^N)$
- nonlinear term $f \in C^1(\mathbb{R})$ with $f(0) = 0$

Then there exists time $T = T(u_0) > 0$, and a unique solution

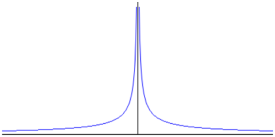
$$u \in L^\infty\left(0, T; L^\infty(\mathbb{R}^N)\right)$$

of equation (1)

(see e.g. Ladyzhenskaya, Solonnikov, Ural'tseva, 1968)

What happens if $u_0 \notin L^\infty(\mathbb{R}^N)$?

For example u_0 singular function:



First results with singular initial data:

due to Weissler (1980-81), Brezis - Cazenave (1996).

They considered: Cauchy problem with power nonlinearities

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1} u & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{2}$$

with $1 < p < \infty$ and with initial data in the Lebesgue spaces $L^q(\mathbb{R}^N)$

For such power type nonlinearities

scale invariance property

plays essential role:

If $u(t,x)$ satisfies (2), then for any $\lambda > 0$ the scaled function

$$u_\lambda = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

also satisfies (2).

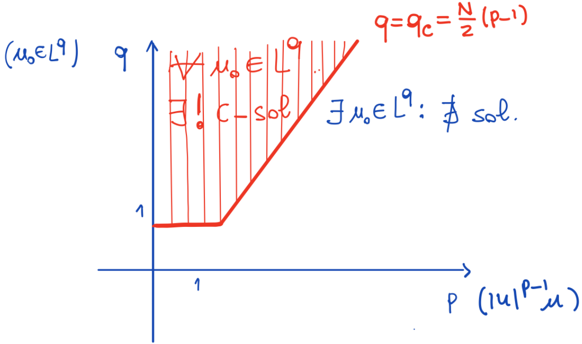
Moreover, the L^q norm is invariant under this scaling iff

$$\boxed{q = q_c := \frac{N(p-1)}{2}} \quad \text{critical exponent}$$

Can classify **existence and uniqueness results** for equation (2) with respect to this critical exponent.

Existence and uniqueness of classical solutions

u is L^q-classical solution $\iff$ $u \in C([0,T), L^q) \cap L^\infty_{loc}((0,T), L^\infty)$, $u(t) \to u_0$, $t \to 0$.



When $q \geq p$, besides classical solutions, one can consider

• u is a **mild solution** $\iff \begin{cases} u \in C\left([0,T);L^q\right) \\ u(t) = \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}|u|^{p-1}u(s)\;ds \end{cases}$

Recall $\mathrm{e}^{t\Delta}u_0$ is the solution of the linear heat equation with initial data u_0.

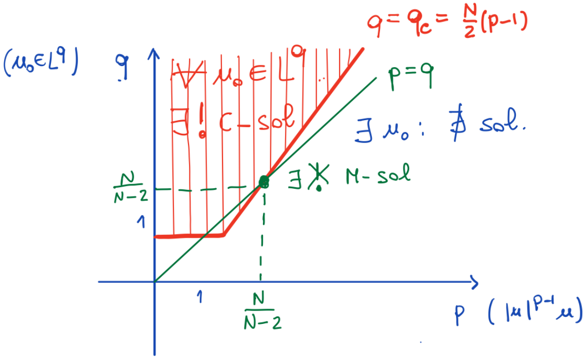
Remarks:

• Any classical solution is a mild solution. Therefore, for $q \geq p$ and $q \geq q_c = \frac{N}{2}(p-1)$ there exists, at least, a mild solution.

• However, uniqueness of mild solutions in the class $C\left([0,T];L^q\right)$ fails for $p = q = q_c$ and in $N \geq 3$ this implies $p = q = \frac{N}{N-2}$

Existence and uniqueness of mild solutions

$$u \text{ is a \textbf{mild solution}} \iff \begin{cases} u \in C([0,T); L^q) \\ u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} |u|^p u(s)\, ds \end{cases}$$



Non-uniqueness for $p = q = \frac{N}{N-2}$, $N \geq 3$

- Ni-Sacks (1985): non-uniqueness for some $u_0 \in L^{\frac{N}{N-2}}(B_1(0))$, $B_1(0) \subset \mathbb{R}^N$:
  - prove **existence of singular solutions** $u_0 \in L^{\frac{N}{N-2}}(B_1)$ to

    $$-\Delta u = u^{\frac{N}{N-2}}, \; u > 0, \; u|_{\partial B_1} = 0$$

    with $u_0(x) \sim |x|^{-\frac{n}{n-2}}(-\log|x|)^{-\frac{n-2}{2}}$, as $x \to 0$

    $\to u_s(t,x) = u_0(x)$ is singular stationary solution to (2)
  - beside the singular stationary solution u_s, by Weissler results a second classical solution $u_r(t,x)$ with initial data u_0 exists.

- T. (2002): extension to the whole space $\mathbb{R}^N$ for suitable singular data u_0 (one solution $u(t)$ remains singular, second solution gets smoothed)

Dimension $N = 2$

In dimension $N = 2$

$$\boxed{p = q = \frac{N}{N-2}}$$

becomes infinite.

Indeed, in dimension $N = 2$ the threshold value $q_c = p - 1$ and for any power nonlinearity $|u|^{p-1}u$, with $p > 1$, the Cauchy problem is wellposed in $L^q(\mathbb{R}^N)$, for any $q \geq q_c = p - 1$ and $q > 1$.

Question: do similar non-uniqueness phenomena also occur in two dimensions ?

One must consider nonlinearities with higher growth than any powers:

exponential nonlinearities

Known results for specific exponential nonlinearities in $\mathbb{R}^2$

First result by

- Ibrahim-Jrad-Majdoub-Saanouni (Bull. Belg. Math. Soc. 2014):

They consider the following heat equation in $\mathbb{R}^2$: critical in regard to Trudinger embedding

$$\begin{cases} \partial_t u - \Delta u = u(e^{u^2} - 1) & \text{in } (0, \infty) \times \mathbb{R}^2 \\ u(0, x) = u_0(x) \end{cases} \tag{3}$$

where $u_0(x) \in H^1(\mathbb{R}^2)$.

They prove : **local existence and uniqueness** for any $u_0 \in H^1(\mathbb{R}^2)$.

Similar results for Schrödinger equation by Nakamura-Ozawa (1998), Colliander-Ibrahim-Majdoub-Masmoudi (2009)...

But one may expect that

heat equation (3) *can be solved in spaces defined by an integrability condition such as Lebesgue spaces or Orlicz spaces.*

Trudinger embedding into Orlicz space:

$$H^1(\Omega) \subset L_\Phi(\Omega), \quad \Omega \subseteq \mathbb{R}^2.$$

where $\Phi(s) = \mathrm{e}^{s^2} - 1$, *highest possible growth* and where the Orlicz space $L_\Phi =: \exp L^2$ is defined as

$$\exp L^2(\mathbb{R}^2) = \left\{ u \in L^1_{loc}(\mathbb{R}^2) \,\Big|\, \int_{\mathbb{R}^2} \left(e^{\left(\frac{|u(x)|}{\lambda}\right)^2} - 1 \right) dx < \infty \text{ for some } \lambda > 0 \right\}$$

and with norm of Luxemburg type

$$\|u\|_{\exp L^2} := \inf \left\{ \lambda > 0; \int_{\mathbb{R}^2} \left(e^{\left(\frac{|u(x)|}{\lambda}\right)^2} - 1 \right) dx \leq 1 \right\}$$

Remarks:

- $\exp L^2(\mathbb{R}^2) \subsetneq L^q(\mathbb{R}^2), \quad \text{for} \quad 2 \leq q < \infty.$
- $\exp L^2(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2).$

A "typical" example of function **not bounded** that belongs to $\exp L^2(\mathbb{R}^2)$

$$u(x) := \begin{cases} (-\log|x|)^{\frac{1}{2}}, & |x| < 1 \\ 0 \ , & |x| \geq 1 \end{cases}$$

In fact:

$$\int_{\mathbb{R}^2} \left(e^{\left(\frac{|u|}{\lambda}\right)^2} - 1\right) dx = 2\pi \int_0^1 \left(e^{\frac{1}{\lambda^2}\left(\log \frac{1}{r}\right)} - 1\right) r\, dr = 2\pi \int_0^1 \left(\frac{1}{r^{\frac{1}{\lambda^2}}} - 1\right) r\, dr$$

which is finite for $\lambda^2 > \frac{1}{2}$ (hence $u \in \exp L^2$).

Note also that

$$u \in \exp L^2(\mathbb{R}^2) \setminus H^1(\mathbb{R}^2)$$

Indeed

$$\int_{B_1(0)} |\nabla u|^2 dx = 2\pi \int_0^1 |u_r|^2 r\, dr = \frac{\pi}{2} \int_0^1 \frac{1}{|\log r| r^2} r\, dr = +\infty.$$

Thus, one can consider the problem (3) in the larger space

$$\exp L^2(\mathbb{R}^2)$$

Indeed, in

- ▷ Ruf.-T. (2002): local existence result for small data $u_0 \in \exp L^2(\mathbb{R}^2)$
- ▷ Ioku (2011): global existence result for small data $u_0 \in \exp L^2(\mathbb{R}^2)$ for equation (3), i.e.

$$\begin{cases} \partial_t u - \Delta u = u(e^{u^2} - 1) & \text{in } (0, \infty) \times \mathbb{R}^2 \\ \quad u(0, x) = u_0(x) & \text{in } \mathbb{R}^2 \end{cases}$$

where $u_0 \in \exp L^2(\mathbb{R}^2)$ small

What about $u_0 \in \exp L^2(\mathbb{R}^2)$ large ?

Indeed, we were able to give a quite complete answer to this question for a **specific** equation.

We provide an explicit nonlinearity

$$f : \mathbb{R}^+ \to \mathbb{R}^+, \quad f(s) \sim \mathrm{e}^{s^2}$$

and an (almost) explicit singular data

$$u_0(x) \in expL^2(B_\rho) \setminus H^1(B_\rho)$$

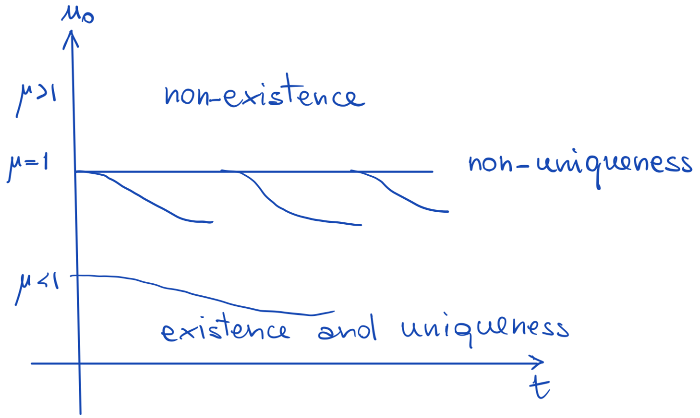
such that for the equation

$$\begin{cases} \partial_t u - \Delta u = f(u) \ \text{ in } B_\rho, \quad\quad u(x,t) = 0 \ \text{ on } \partial B_\rho \\ \quad u(0,x) = u_0(x) \end{cases} \tag{4}$$

Theorem (Ioku, Ruf, T. Ann. Inst. Poincaré, 2019)

The following **trichotomy** *holds*

- *for* μu_0 *with* $0 < \mu < 1$*: exists unique solution*
- *for* u_0*: nonuniqueness (singular stationary and regularizing solution)*
- *for* μu_0 *with* $\mu > 1$*: exists no solutions*



How are $f(u)$ and the (almost) explicit singular data u_0 defined ?

- *An explicit singular solution*

  Let

$$w(|x|) = \sqrt{-2 \log |x|}$$

  Then

$$w \in expL^2(B_1) \setminus H^1(B_1)$$

  One calculates that $w(|x|)$ is explicit solution to the equation

$$-\Delta u = \frac{1}{u^3} e^{u^2} \;,\; u(|x|) > 0 \tag{5}$$

  or, equivalently, of

$$-u'' - \frac{1}{r} u' = \frac{1}{u^3} e^{u^2} \;, \tag{6}$$

  (de Figueiredo - Ruf, CPAM, 1995)

- *The nonlinearity*

The nonlinearity $f(u) = \frac{1}{u^3}\, e^{u^2}$ is singular in the origin.

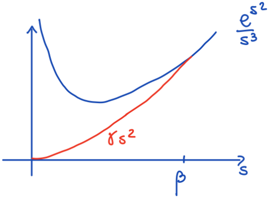
We need a nonlinearity with $f(0) = 0$:

so let the nonlinearity $f(s)$ be given as follows

$$f(s) := \begin{cases} \frac{1}{s^3}\, e^{s^2}\,, & \text{if } \; s \geq \beta \\ \gamma s^2\,, & \text{if } \; 0 \leq s \leq \beta \end{cases}$$

with $\gamma = (\frac{2e}{5})^{5/2}$ and $\beta = \sqrt{\frac{5}{2}}$.

This function f is $C^1([0, +\infty)$, increasing and convex, $f(0) = f'(0) = 0$.



$$f(s) = \begin{cases} \frac{e^{s^2}}{s^3} & s > \beta \\ \gamma s^2 & 0 \leq s \leq \beta \end{cases}$$

$\beta = \sqrt{\frac{5}{2}} \quad \gamma = \left(\frac{2e}{5}\right)^{5/2}$

$f \in C^1([0, +\infty)),\; f(0) = f'(0) = 0$

- *The almost explicit singular solution*

Note that the function

$$w(r) = \sqrt{-2 \log r}$$

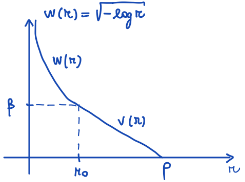
satisfies

$$w(r) \geq \beta \iff r \leq r_0$$

Let us consider the following initial value problem

$$\begin{cases} -v'' - \frac{1}{r}v' = \gamma v^2, \quad r \in [r_0, +\infty) \\ v(r_0) = \beta = w(r_0) \\ v'(r_0) = w'(r_0) \end{cases} \tag{7}$$

Now show using sooting method:
there exists a first zero $\rho > r_0$ of the solution $v(r)$ of equation (7)



Then

$$u_0(x) = \begin{cases} w(|x|), & 0 < |x| < r_0 \\ v(|x|), & r_0 \le |x| < \rho \end{cases}$$

solves equation

$$-u'' - \frac{1}{r}u' = f(u) \tag{8}$$

on $(0, \rho)$, with $u_0(\rho) = 0$.

Finally show that the *singular* solution u_0 satisfies the elliptic equation

$$-\Delta u = f(u) \ \text{ in } \ B_\rho(0) \ , \ u|_{\partial B_\rho} = 0$$

in the *distributional sense* in $B_\rho(0)$:

$$\int_{B_\rho(0)} u_0 \Delta\varphi + f(u_0)\varphi f(u_0) dx = 0$$

for all $\varphi \in C_0^\infty(B_\rho(0))$

With this singular solution u_0 one proves theTheorem

A related result

- Ibrahim-Kikuchi-Nakanishi-Wei (Math. Ann. 2021):

  They consider the nonlinearity

$$f(u) = ue^{u^2}$$

  prove existence of a singular solution to $-\Delta u = f(u)$

  with $u_0(x) = \sqrt{-2\log|x| - 2\log(-2\log|x|)} + O\left(\frac{\log(-\log|x|)}{\sqrt{-\log|x|}}\right)$, as $x \to 0$

  and obtain a similar non-uniqueness result in $\mathbb{R}^2$

Generalizations

- Can one obtain singular solutions for any $f(u)$ of the form:

$$f(u) = u^r \mathrm{e}^{u^2}, \quad r \in \mathbb{R}?$$

Actually, more is true !

Theorem (Fujishima-Ioku-Ruf-T., 2024 - Singular solutions)

1) For $q > 1$ and $r \in \mathbb{R}$, there exists a singular solution U (locally around the origin) of

$$-\Delta u = u^r \mathrm{e}^{u^q}$$

satisfying

$$U(x) = \left(\log \frac{1}{|x|^2} - \frac{2q + r - 1}{q} \log \log \frac{1}{|x|^2} + \log \frac{4(q-1)}{q^2} \right)^{\frac{1}{q}} + O\left(\frac{\log(-\log|x|)}{(-\log|x|)^{2-\frac{1}{q}}} \right), \quad \text{as} \quad x \to 0$$

Theorem (Fujishima-Ioku-Ruf-T., 2024 - Singular solutions)

2) For $-\Delta u = e^{e^u}$ *there exists a singular solution* U *(locally around the origin) satisfying*

$$U(x) = \log\left[-\log w - \log(-\log w)\right] + O\left(\frac{\log(-\log w)}{(-\log w)^2}\right),$$

as $x \to 0$, *where*

$$w = \frac{1}{4}|x|^2\left(\log\frac{1}{|x|^2} + 1\right).$$

Remarks

• Our result contains the previous ones of Ioku-Ruf-T and Ibrahim-Kikuchi-Nakanishi-Wei respectively for $r = -3$, $q = 2$ and for $r = 1$, $q = 2$.

• The same result holds also for any $f \in C^1([0, +\infty))$, with $f(u) = u^r e^{u^q}$, with $r \in \mathbb{R}$ and $q > 1$ or $f(u) = e^{e^u}$ for large value of u.

• A similar result can be obtained for a large class of nonlinearity. For example we are able to find singular solution for nonlinearities $f(u) = e^{u^q + u^r}$, $q > 1$, $q > 2r$, $r > 0$.

However we are not able to deal with $f(u) = e^{u^q (\log u)^r}$, $q > 1$, $r \in \mathbb{R}$, $r \neq 0$.

There is no general method for finding singular solutions !

However, we have a family of

model exponential type nonlinearities with explicit singular solutions (N=2)

Indeed, following the example of de Figueiredo-Ruf we observe :

$$v(x) = (-2\log|x|)^{\frac{1}{q}} : \quad \text{singular solution to}$$
$$-\Delta v = \frac{4}{qq'}\frac{1}{v^{2q-1}}\mathrm{e}^{v^q}, \quad q > 1$$

$$v(x) = \log(-2\log|x|) : \quad \text{singular solution to}$$
$$-\Delta v = 4\frac{1}{\mathrm{e}^{2v}}\mathrm{e}^{\mathrm{e}^v}$$

Proof of Theorem - Singular solutions

The proof relies on a general theorem, which deals with more general nonlinearities and whose statement is quite complicated.

The main idea is:

- Let $f(u) = u^r \mathrm{e}^{u^q}$, with $q > 1$ be a given nonlinearity.

Associate to f a model nonlinearity g which is *close* to f. In particular if

$$F(s) = \int_s^\infty \frac{1}{f(\eta)} d\eta < \infty, \quad s > 0 \text{ and } G(s) = \int_s^\infty \frac{1}{g(\eta)} d\eta < \infty, \quad s > 0$$

then

$$\lim_{s\to\infty} f'(s)F(s) = \lim_{s\to\infty} g'(s)G(s) = 1$$

with the *same rate* of convergence.

- Fujishima-Ioku (2018): if the function v satisfies $-\Delta v = g(v)$ then the function

$$\tilde{u} := F^{-1}[G(v)]$$

satisfies

$$-\Delta \tilde{u} = f(\tilde{u}) + \frac{|\nabla \tilde{u}|^2}{f(\tilde{u})F(\tilde{u})}\left[f'(\tilde{u})F(\tilde{u}) - g'(v)G(v)\right]$$

Therefore, we guessed that it should exist a solution u of $-\Delta u = f(u)$ close to $\tilde{u}$ if the remainder term is small.

- Show: there exists small $\theta(x)$ such that

$$U(x) = \tilde{u}(x) + \theta(x)$$

is singular and solves $-\Delta u = f(u)$.

Non-uniqueness result for general $f(u)$

Let $f \in C^1([0,\infty)$ with $f(u) = u^r e^{u^q}$, $q > 1$ $r \in \mathbb{R}$ or $f(u) = e^{e^u}$, for large value of u.

By shooting method one can continue this singular solution to a solution u_0 of

$$-\Delta u = f(u), \quad u|_{\partial B_R} = 0$$

such that $u_0(x) \sim U(x)$ as $x \to 0$.

Using this singular solution u_0 as initial data, one gets nonuniqueness

Theorem (Fujishima, Ioku, Ruf, T. 2024 - Non-uniqueness)

Let $f \in C^1([0,\infty)$ with $f(u) = u^r e^{u^q}$, $q > 1$ $r \in \mathbb{R}$ or $f(u) = e^{e^u}$, for large value of u.

For the equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(u)\,, & \text{in } (0,T) \times B_R, \quad u|_{\partial B_R} = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{9}$$

one has nonuniqueness : there exist a singular stationary solution and a regularizing solution.

Thank you for your attention !

Happy birthday Pierre Gilles !