# Large self-similar solutions to Oberbeck–Boussinesq System with Newtonian gravitational field

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# Fonctions d'échelle interpolantes, polynômes de Bernstein et ondelettes non stationnaires

**Pierre Gilles Lemarié-Rieusset**

**Résumé.** La théorie de la convergence des fonctions d'échelle (non-stationnaires) et l'approximation des filtres d'échelle interpolants à l'aide de polynômes de Bernstein, permettent la construction d'une fonction d'échelle interpolante non-stationnaire aux propriétés d'approximation remarquables.

1. Self-similar solutions of the Navier–Stokes equations. A quick review

2. The Boussinesq system
   - Setting of the problem and statement of the Theorem
   - Analysis of the perturbed system in a bounded domain
   - Existence of solutions of the perturbed elliptic system in the whole space

## Navier–Stokes

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u, \\ \nabla \cdot u = 0, \end{cases} \qquad x \in \mathbb{R}^3, \ t \in \mathbb{R}_+ \tag{NS}$$

Unknowns:

- fluid velocity $u = u(x,t)$, $u = (u_1, u_2, u_3)$.
- pressure $p = p(x,t)$.

**Self-similar solutions:** $u(x,t) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$ and $p(x,t) = \frac{1}{t} P(\frac{x}{\sqrt{t}})$.
Equivalently,

$$\forall \lambda > 0 \text{ and all } (x,t) \in \mathbb{R}^3 \times (0,\infty): \quad \begin{aligned} u(x,t) &= \lambda u(\lambda x, \lambda^2 t) \\ p(x,t) &= \lambda^2 p(\lambda x, \lambda^2 t). \end{aligned}$$

Early construction, using the vorticity formulation: Giga-Miyakawa (1989)

## Cannone-Meyer-Planchon's general strategy (1993)

- Choose a space $X$ (for initial data) of tempered distributions, such that $\|\lambda u_0(\lambda\cdot)\|_X = \|u_0\|_X$, for all $\lambda > 0$ and all $u_0 \in X$.
- Choose a function space $\mathcal{E}$ (for solutions) such that $t \mapsto e^{t\Delta}u_0 \in \mathcal{E}$ and

$$\|t \mapsto e^{t\Delta}u_0\|_{\mathcal{E}} \lesssim \|u_0\|_X \qquad \text{and} \qquad \|B(u,v)\|_{\mathcal{E}} \lesssim \|u\|_{\mathcal{E}}\|v\|_{\mathcal{E}}.$$

  where

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(s)\,\mathrm{d}s.$$

  Then the map $u \mapsto e^{t\Delta}u_0 - B(u,u)$ is contractive in a ball $\{u\colon \|u\|_{\mathcal{E}} < \eta\}$, for $\eta > 0$ small enough.
- Choose a datum $u_0 \in X$, solenoidal and homogeneous of degree $-1$, such that

$$\|u_0\|_X \lesssim \eta.$$

  The fixed point of the contraction is the unique solution $u$ of NS in $\mathcal{E}$ of small norm, starting from $u_0$. Moreover,

$$\boxed{\forall\lambda > 0,\ u_0(x) = \lambda u_0(\lambda x)} \Rightarrow \boxed{\forall\lambda > 0,\ u(x,t) = \lambda u(\lambda x, \lambda^2 t)}$$

A very basic example is

$$u_0(x) = \eta\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}, 0\Big), \quad \eta > 0 \text{ small enough.}$$

This datum gives rise to a self-similar solution of NS: the above strategy applies and several functional settings are possible.

For instance,

1. $\|v_0\|_X := \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |x|\,|v_0(x)|$ and $\|v\|_{\mathcal{E}} := \operatorname{ess\,sup}_{x,t}(|x| + \sqrt{t})|v(x,t)|$.
(Cannone, Meyer, Planchon)
2. $X = L^{3,\infty}(\mathbb{R}^3)$, and $\mathcal{E} = C_w([0,\infty), L^{3,\infty}(\mathbb{R}^3))$.
(Barraza)
3. $X = \dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ and $\|v\|_{\mathcal{E}} := \sup_{t>0} t^{\frac{1}{2}(1-3/p)}\|v(t)\|_{L^p}$.
(Cannone)
4. $X = BMO^{-1}$ and $\mathcal{E}$ the Koch-Tataru space

The common feature of these spaces is the scaling. The inclusions for $X$ are

$$(1) \subset (2) \subset (3) \subset (4).$$

**Interest of choosing a "small" space $X$**

- More precise description of the self-similar solution.
  For example, for small solutions constructed in the space (1) we have, as $|x| \to +\infty$.

**(L.B. ARMA, 2009)**

$$U(x) = \underbrace{u_0(x)}_{\approx |x|^{-1}} + \underbrace{\Delta u_0(x) - \mathbb{P}\nabla\cdot(u_0 \otimes u_0)(x)}_{\approx |x|^{-3}} - \underbrace{\frac{Q(x)}{|x|^7}A(u_0)}_{\approx |x|^{-4}} + O\big(|x|^{-5}\log|x|\big)$$

*where $A(u_0)$ is a constant matrix and the components of $Q(x)$ are explicitly known homogeneous polynomials.*

**Interest of choosing a "large" space $X$**

- It allows to consider more singular data
- Smallness condition put on a weaker norm

## A motivation

Selfsimilar solutions describe the behavior as $t \to \infty$ of a large class of solutions :

If $u_0$ and $u_{0,H}$ are two small data and $u_{0,H}$ is homogeneous,

$$\begin{array}{rcccl} \text{If} & u_0 & \approx & u_{0,\mathrm{H}} & \text{as } |x| \to \infty \\ \text{then} & u & \approx & \frac{1}{\sqrt{t}} U(\frac{\cdot}{\sqrt{t}}) & \text{as } t \to \infty \end{array}$$

Resultats in this direction (Planchon 1998, Cazenave, Dickstein, Weissler 2005):

$$\|u(\cdot,t) - \tfrac{1}{\sqrt{t}} U(\tfrac{\cdot}{\sqrt{t}})\|_p = o(t^{-\frac{1}{2}(1-3/p)}),$$
$$\|\,|\cdot|\,\big(u(\cdot,t) - \tfrac{1}{\sqrt{t}} U(\tfrac{\cdot}{\sqrt{t}})\big)\|_\infty = o(1) \quad \text{as } t \to \infty.$$

## Removing the smalless condition

The problem of constructing **large self-similar solutions** of NS remained open for 25 years.

Two difficulties:

- One needs to look for **global large solutions**. This requires going beyond classical Leray $L^2$-theory, as homogeneous initial data of degree $-1$ are not in $L^2$.
- In the lack of appropriate uniqueness theorem, not all solutions arising from homogeneous data are necessary self-similar.

But:

- P.G. Lemarié-Rieusset theory (C.R. Acad. Sci. Paris, 1999) of $L^2_{uloc}$-solutions provide the appropriate functional framework.
- Jia & Sverák (Inventiones, 2014) use this framework and establish subtle a priori Hölder estimates for $L^2_{uloc}$ self-similar solutions arising from homogeneous data with Hölder regularity outside 0. Applying Leray-Schauder theorem they deduced the existence of large self-similar solutions.

P.G. Lemarié-Rieusset (2016) considerably relaxed the regularity condition on the data:

**Theorem (Chapter 16 of his book "The NS problem in the 21th Century")**

*Let $u_0 \in L^2_{loc}(\mathbb{R}^3)$, divergence-free and homogeneous of degree $-1$.*

*Then there exists a least one self-similar solution of NS,*

$$u(x,t) = \frac{1}{\sqrt{t}} U(x/\sqrt{t}),$$

*such that $U \in H^1_{loc}$ and $u(t) \to u_0$ as $t \to 0$ strongly in $L^2_{loc}$.*

The key observations are that such a $u_0$ must be $L^2_{uloc}$ and can be approximated by a sequence $u_{0,k}$ of data bounded on the sphere. A construction similar to Jia& Sverák's provides a self-similar solution $u_k$ arising from $u_{0,k}$ satisfying an a priori bound depending only on the $L^2_{uloc}$ norm of their initial data and converging to a self-similar solutions arising from $u_0$.

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## The Boussinesq system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + \theta \nabla G + f, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta = \Delta \theta, \end{cases} \qquad x \in \mathbb{R}^3, \ t \in \mathbb{R}_+ \qquad \text{(B)}$$

Unknowns:

- fluid velocity $u = u(x,t)$, $u = (u_1, u_2, u_3)$.
- pressure $p = p(x,t)$.
- temperature variations from an equilibrium: $\theta = \theta(x,t)$

Given quantities:

- $f = f(x,t)$: external force.
- $G$ is the gravitational potential. Here $\theta \nabla G$ denotes the buoyancy force, proportional to the temperature variations and to the gravitational force acting on the fluid.

We will adress the construction of **forward large self-similar solutions**, under appropriate assumptions on $\nabla G$ and $f$.

## Which choice for the gravitational force ?

In bounded domains a reasonable choice is $\nabla G = g[0,0,-1]$ (with $g$ constant).

In the whole $\mathbb{R}^3$ it is still common to take $\nabla G = g[0,0,-1]$ (with $g$ constant) but this choice is **no longer physically realistic**. Indeed:

**Theorem (with M. Schonbek and C. Mouzouni, 2016)**

*Let* $\nabla G = g[0,0,-1]$ *and assume that*

- $f \equiv 0$.
- $u_0 \equiv 0$
- $\theta_0 \in L^1(\mathbb{R}^3)$, *with small* $L^1(\mathbb{R}^3)$*-norm.*

*then*

$$\left(\int \theta_0\right)\sqrt{t} \lesssim \|u(t)\|_{L^2}^2 \lesssim \left(\int \theta_0\right)\sqrt{t} \qquad \textit{for large time.}$$

In particular, if $\int \theta_0 \neq 0$, then the **energy grows large** as $t \to +\infty$.

## Choosing the appropriate potential $G$

- Let us neglect the self-gravitation of fluid particles. We take

$$G(x) = -\int_{\mathbb{R}^3} \frac{1}{|x-y|} m(y)\, dy,$$

where $m$ denotes the mass density of the object acting on the fluid by means of gravitation.

- Assume the size of the object is negligible: one is led to choosing

$$\boxed{G(x) = -|x|^{-1}}.$$

The Boussinesq system (B) with the above gravitational potential was rigorously derived by E. Feireisl and M. Schonbek as a singular limit of the full Navier-Stokes–Fourier system with suitable boundary conditions and with

- the Mach and the Froude numbers tending to zero, and
- when the family of domains on which the primitive problems are stated converges to the whole space $\mathbb{R}^3$.

With this choice of $G$, the temperature and the velocity enjoy the same scaling.

A *self-similar solution* to system (B) is, by definition, a solution that can be written in the form

$$u(x,t) = \frac{1}{\sqrt{2t}} U\Big(\frac{x}{\sqrt{2t}}\Big), \qquad \theta(x,t) = \frac{1}{\sqrt{2t}} \Theta\Big(\frac{x}{\sqrt{2t}}\Big),$$

with $U(x) = u(x, 1/2)$ and $\Theta(x) = \theta(x, 1/2)$.

Our main result is the following
(inspired by Korobkov & Tsai, Analysis and PDE 2016):

## Theorem (with Grzegorz Karch)

*Let* $(u_0, \theta_0) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^3 \backslash \{0\})$ *be homogeneous of degree* $-1$, *with* $\nabla \cdot u_0 = 0$. *Let the external force* $f(x,t)$ *be of the form*

$$f(x,t) = \frac{1}{(\sqrt{2t})^3} F\left(\frac{x}{\sqrt{2t}}\right).$$

*with the profile* $F \in H^{-1}(\mathbb{R}^3)^3$.

*Then there exists a self-similar solution of system (B). This solution has the following properties:*

- $(u, \theta) \in C_w([0,\infty), \mathbf{L}^{3,\infty}(\mathbb{R}^3))$;
- *for some constants* $c, c' \geq 0$ *and all* $t > 0$,

$$\|u(t) - e^{t\Delta} u_0\|_2 + \|\theta(t) - e^{t\Delta}\theta_0\|_2 = ct^{1/4},$$
$$\|\nabla u(t) - \nabla e^{t\Delta} u_0\|_2 + \|\nabla\theta(t) - \nabla e^{t\Delta}\theta_0\|_2 = c' t^{-1/4}.$$

## Strategy of the proof

- We substitute the expression for self-similar solutions into (B) to eliminate the time variable and study the elliptic system

$$\begin{cases} -\Delta U - U - (x\cdot\nabla)U + (U\cdot\nabla)U + \nabla P = \Theta\,\nabla(|\cdot|^{-1}) + F, \\ \nabla\cdot U = 0, \\ -\Delta\Theta - \Theta - (x\cdot\nabla)\Theta + \nabla(\Theta U) = 0. \end{cases} \tag{S}$$

- We take advantage of the fact that the solution to the heat equation with the same initial data $(u_0, \theta_0)$ is itself of self-similar form:

$$e^{t\Delta}u_0(x) = \frac{1}{\sqrt{2t}}U_0\left(\frac{x}{\sqrt{2t}}\right) \quad \text{and} \quad e^{t\Delta}\theta_0(x) = \frac{1}{\sqrt{2t}}\Theta_0\left(\frac{x}{\sqrt{2t}}\right),$$

with self-similar profiles

$$U_0 := e^{\Delta/2}u_0 \quad \text{and} \quad \Theta_0 := e^{\Delta/2}\theta_0$$

- Rather than studying directly system (S), we study a perturbated system, for the new unknowns

$$V = U - U_0 \quad \text{and} \quad \Psi = \Theta - \Theta_0.$$

The new elliptic system for the perturbated quantities reads

$$\begin{cases} -\Delta V - V - (x\cdot\nabla)V + (V+U_0)\cdot\nabla(V+U_0) + \nabla P = (\Psi+\Theta_0)\,\nabla(|\cdot|^{-1}) + F, \\ \nabla\cdot V = 0, \\ -\Delta\Psi - \Psi - (x\cdot\nabla)\Psi + \nabla\cdot\big((\Psi+\Theta_0)(V+U_0)\big) = 0. \end{cases} \tag{PS}$$

We will construct solutions of to the **perturbed system** (PS) in the Sobolev space $H^1(\mathbb{R}^3)^4$.

Previous strategy is similar to that of Korobkov-Tsai paper (2016) for the unforced NS equation. Their paper corresponds to the case

$$\Psi \equiv \Theta_0 \equiv F \equiv 0.$$

## Analysis of the perturbed system in a bounded domain

Let

- $\Omega$ a bounded domain in $\mathbb{R}^3$ with smooth boundary.
- $\mathbf{H}(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)^3 \times C^\infty_0(\Omega)$ in the Sobolev space $H^1(\Omega)^4$.
- $\rho \in C_b(\mathbb{R}^3)$ (a cut off function that will be used to smooth out the singularity of $|\cdot|^{-1}$).
- $\lambda \in [0,1]$ a parameter.

We start considering the system in $\Omega$

$$\begin{cases} -\Delta V + \nabla P = \lambda\Big(V + x\cdot\nabla V - (V+U_0)\cdot\nabla(V+U_0) + (\Psi+\Theta_0)\,\rho\nabla(|\cdot|^{-1}) + F\Big), \\ -\Delta\Psi = \lambda\Big(\Psi + x\cdot\nabla\Psi - \nabla\cdot\big((\Psi+\Theta_0)(V+U_0)\big)\Big), \\ \nabla\cdot V = 0, \\ V = \Psi = 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{$\lambda$-PS}$$

## The key a priori estimate

### Proposition

*Assume that* $F \in H^{-1}(\Omega)^3$. *Let* $(V, \Psi) \in \mathbf{H}(\Omega)$ *be a solution to problem* ($\lambda$-PS). *There exists a constant* $C_0 = C_0(\Omega, F, \rho, U_0, \Theta_0)$, *independent on* $\lambda \in [0,1]$, *such that*

$$\int_\Omega \left(|V|^2 + \Psi^2 + |\nabla V|^2 + |\nabla \Psi|^2\right) \leq C_0. \qquad (1)$$

**Proof.**
Multiplying first equation of system ($\lambda$-PS) by $V$, the second by $\Psi$, after some integration by parts, using cancellations like $\int_\Omega \nabla\cdot(\Psi(V+U_0))\Psi = 0$, we get

$$\begin{aligned}\int_\Omega |\nabla V|^2 + \frac{\lambda}{2}\int_\Omega |V|^2 =&\lambda\Big[-\int_\Omega (U_0\cdot\nabla U_0)\cdot V - \int_\Omega (V\cdot\nabla U_0)\cdot V \\ &+ \int_\Omega \Big((\Psi+\Theta_0)\rho\nabla|\cdot|^{-1}\Big)\cdot V + \langle F, V\rangle\Big] \qquad (2)\\ \int_\Omega |\nabla\Psi|^2 + \frac{\lambda}{2}\int_\Omega \Psi^2 + \lambda\int_\Omega \nabla\cdot[\Theta_0(V+U_0)]\Psi = 0.\end{aligned}$$

All the integrals are convergent, because of the nice properties of $(U_0,\Theta_0) = e^{\Delta/2}(u_0,\theta_0)$:

$$|(U_0,\Theta_0)| \le C(1+|x|)^{-1}, \qquad |\nabla(U_0,\Theta_0)| \le C(1+|x|)^{-2}.$$

**Difficulty.** $U_0$ and $\Theta_0$ can be **large**: not obvious to absorb in the LFS the integrals containing these terms.

Estimating the latter integral we obtain, for $0 \le \lambda \le 1$,

$$\frac{1}{2}\int_\Omega |\nabla \Psi|^2 + \frac{\lambda}{2}\int_\Omega \Psi^2 \le \lambda\Big(\|\Theta_0\|_\infty^2 \int_\Omega |V|^2 + \int_\Omega |\Theta_0 U_0|^2\Big). \qquad (3)$$

So, to obtain the a priori estimate of the proposition, it is sufficient to prove that

$$\int_\Omega |\nabla V|^2 \le C_1 \qquad (4)$$

for some $C_1 = C_1(\Omega, F, \rho, U_0, \Theta_0) > 0$ independent on $\lambda \in [0,1]$.

We argue by **contradiction**.

We assume that there exist a sequence $(\lambda_k) \subset [0,1]$ and a sequence of solutions $(V_k, \Psi_k) \subset \mathbf{H}(\Omega)$ to problem ($\lambda$-PS), such that

$$\Big(\int_\Omega |\nabla V_k|^2\Big)^{1/2} \to +\infty.$$

Let us set

$$J_k := \left( \int_\Omega |\nabla V_k|^2 \right)^{1/2} \to +\infty, \qquad L_k := \left( \int_\Omega |\nabla \Psi_k|^2 \right)^{1/2}.$$

*Step 1. Excluding the case:* $\limsup_{k\to+\infty} J_k/L_k < \infty$.

We introduce the normalized functions

$$\widehat{V}_k = \frac{V_k}{J_k} \quad \text{and} \quad \widehat{\Psi}_k = \frac{\Psi_k}{L_k},$$

so that $(\widehat{V}_k, \widehat{\Psi}_k)$ is a bounded sequence in $\mathbf{H}(\Omega)$. After extraction:

- $(\widehat{V}_k, \widehat{\Psi}_k) \to (\widetilde{V}, \widetilde{\Psi})$ weakly in $\mathbf{H}(\Omega)$ and strongly in $\mathrm{L}^p(\Omega)$, for $p \in [2, 6)$.
- $\lambda_k \to \lambda_0$, for some $\lambda_0 \in [0, 1]$.

If by contradiction, $\limsup_{k\to+\infty} J_k/L_k < \infty$, then after a new extraction of a subsequence, we can assume that there exists $\gamma \geq 0$ such that

$$J_k/L_k \to \gamma.$$

In fact, $\gamma > 0$ by estimate (3).
Moreover, as $J_k \to +\infty$, we must have $L_k \to +\infty$.

Dividing by $L_k^2$ the energy equality for $\Psi_k$ we get,

$$\underbrace{\frac{1}{L_k^2}\int_\Omega |\nabla\Psi_k|^2}_{=1} + \underbrace{\frac{\lambda_k}{2L_k^2}\int_\Omega |\Psi_k|^2}_{\to \frac{\lambda_0}{2}\int_\Omega |\widetilde{\Psi}|^2} + \underbrace{\frac{\lambda_k}{L_k^2}\int_\Omega \nabla\cdot(\Theta_0 V_k)\Psi_k}_{\to\ldots} + \underbrace{\frac{\lambda_k}{L_k^2}\int_\Omega \nabla\cdot(\Theta_0 U_0)\Psi_k}_{\to 0} = 0.$$

Here we use

$$\frac{\lambda_k}{L_k^2}\int_\Omega \nabla\cdot(\Theta_0 V_k)\Psi_k = \frac{\lambda_k J_k}{L_k}\int_\Omega \nabla\cdot(\Theta_0 \widehat{V}_k)\widehat{\Psi}_k \to \lambda_0\gamma\int_\Omega \nabla\cdot(\Theta_0\widetilde{V})\widetilde{\Psi}$$

and

$$\frac{1}{L_k^2}\Big|\int_\Omega \nabla\cdot(\Theta_0 U_0)\Psi_k\Big| \le \frac{C}{L_k}\|\Theta_0 U_0\|_{L^2(\Omega)} \to 0.$$

Hence, we get in the limit as $k\to+\infty$, the identity

$$\boxed{1 + \frac{\lambda_0}{2}\int_\Omega |\widetilde{\Psi}|^2 = -\lambda_0\gamma\int_\Omega \nabla\cdot(\Theta_0\widetilde{V})\,\widetilde{\Psi}}$$

The weak formulation of the second equation of ($\lambda$-PS), gives, for all $\chi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla\Psi_k \cdot \nabla\chi = \lambda_k \Big[ \int_\Omega [\Psi_k + x\cdot\nabla\Psi_k]\chi - \int_\Omega \nabla \cdot [(\Psi_k + \Theta_0)(V_k + U_0)]\chi \Big].$$

Hence,

$$\underbrace{\frac{1}{L_k^2}\int_\Omega \nabla\Psi_k \cdot \nabla\chi}_{\to 0} = \underbrace{\frac{\lambda_k}{L_k^2}\int_\Omega [\Psi_k + x\cdot\nabla\Psi_k]\chi}_{\to 0} - \underbrace{\frac{\lambda_k}{L_k^2}\int_\Omega \nabla \cdot [(\Psi_k + \Theta_0)(V_k + U_0)]\chi}_{\to \lambda_0\gamma \int_\Omega \nabla\cdot(\widetilde{\Psi}\widetilde{V})\chi}.$$

Hence, we get in the limit the identity

$$\boxed{\lambda_0\gamma \int_\Omega \nabla\cdot(\widetilde{\Psi}\widetilde{V})\chi = 0} \qquad \text{for all } \chi \in C_0^\infty(\Omega).$$

Let us combine the two identities

$$\begin{cases} 1+\dfrac{\lambda_0}{2}\displaystyle\int_\Omega |\widetilde\Psi|^2 = -\lambda_0\gamma\int_\Omega \nabla\cdot(\Theta_0\widetilde V)\,\widetilde\Psi \\ \lambda_0\gamma\displaystyle\int_\Omega \nabla\cdot(\widetilde\Psi\widetilde V)\chi = 0 \qquad \text{for all } \chi\in C_0^\infty(\Omega) \end{cases}$$

The former implies $\lambda_0\gamma\neq 0$. Then latter implies

$$\int_\Omega \nabla\cdot(\widetilde\Psi\widetilde V)\chi = 0 \qquad \text{for all } \chi\in C_0^\infty(\Omega).$$

But then

$$\widetilde V\cdot\nabla\widetilde\Psi = 0.$$

So,

$$\int_\Omega \nabla\cdot(\Theta_0\widetilde V)\widetilde\Psi = -\int_\Omega \Theta_0\widetilde V\cdot\nabla\widetilde\Psi = 0$$

which contradicts the first identity of our system.

This excludes that $\limsup_{k\to+\infty} J_k/L_k<\infty$.

*Step 2.* We reduced ourselves to the case $\limsup_{k\to+\infty} J_k/L_k = +\infty$.

After extracting a new subsequence, we can assume that $L_k/J_k \to 0$.
The **energy equality** for $V_k$ reads

$$\int_\Omega |\nabla V_k|^2 + \frac{\lambda}{2}\int_\Omega |V_k|^2 = \lambda\Big[-\int_\Omega (U_0\cdot\nabla U_0)\cdot V_k - \int_\Omega (V_k\cdot\nabla U_0)\cdot V_k + \int_\Omega \Big((\Psi_k+\Theta_0)\rho\nabla|\cdot|^{-1}\Big)\cdot V_k + \langle F, V_k\rangle\Big].$$

Let us divide it by $J_k^2$ and study the limit of each term, as $k\to+\infty$. We have

$$\frac{1}{J_k^2}\int_\Omega |\nabla V_k|^2 = 1, \qquad \frac{1}{J_k^2}\int_\Omega |V_k|^2 \to \int_\Omega |\widetilde{V}|^2, \qquad \frac{1}{J_k^2}\int_\Omega (U_0\cdot\nabla U_0)\cdot V_k \to 0,$$

and

$$\frac{1}{J_k^2}\langle F, V_k\rangle \to 0, \qquad \frac{1}{J_k^2}\int_\Omega (V_k\cdot\nabla U_0)\cdot V_k \to \int_\Omega (\widetilde{V}\cdot\nabla U_0)\cdot\widetilde{V},$$

because $|\nabla U_0| \in L^2(\Omega)$ and $\widehat{V}_k \to \widetilde{V}$ strongly in $L^p(\Omega)^3$ for $p\in[2,6)$.
For the blue term we rely on the Hardy inequality:

**Estimating** $\int_\Omega \left((\Psi_k + \Theta_0)\rho\nabla|\cdot|^{-1}\right)\cdot V_k$.

As $\Psi_k$ and $V_k$ belong to $H^1_0(\Omega)$ we can write

$$\begin{aligned}\frac{1}{J_k^2}\Big|\int_\Omega \Psi_k\rho\nabla\left(|\cdot|^{-1}\right)\cdot V_k\Big| &\le \frac{C}{J_k^2}\Big\|\frac{\Psi_k}{|\cdot|}\Big\|_{L^2(\Omega)}\Big\|\frac{V_k}{|\cdot|}\Big\|_{L^2(\Omega)} \\ &\le \frac{C}{J_k^2}\|\nabla\Psi_k\|_{L^2(\Omega)}\|\nabla V_k\|_{L^2(\Omega)} \\ &= \frac{C}{J_k}\|\nabla\Psi_k\|_{L^2(\Omega)} = C\frac{L_k}{J_k} \to 0.\end{aligned} \tag{5}$$

The function $\Theta_0$ does not belong to $H^1_0(\Omega)$. However, $\Theta_0/|\cdot| \in L^2(\mathbb{R}^3)$. Therefore, the term of containing $\Theta_0$ can be estimated as in (5). Namely,

$$\frac{1}{J_k^2}\Big|\int_\Omega \Theta_0\rho\nabla\left(|\cdot|^{-1}\right)\cdot V_k\Big| \le \frac{C}{J_k^2}\Big\|\frac{\Theta_0}{|\cdot|}\Big\|_{L^2(\Omega)}\Big\|\frac{V_k}{|\cdot|}\Big\|_{L^2(\Omega)} \le \frac{C}{J_k} \to 0.$$

The above calculations lead to the identity

$$\boxed{1 + \frac{\lambda_0}{2}\int_\Omega |\widetilde{V}|^2 = -\lambda_0\int_\Omega (\widetilde{V}\cdot\nabla U_0)\cdot\widetilde{V}}$$

In particular, $\lambda_0 \neq 0$, so for large $k$, we have $\lambda_k \neq 0$.

We now write the **weak formulation** of the equation satisfied by $V_k$: after dividing by $\lambda_k J_k^2$, we obtain, for any solenoidal vector field $\eta \in C^\infty_{0,\sigma}(\mathbb{R}^3)^3$,

$$\begin{aligned}
&\frac{1}{\lambda_k J_k^2}\int_\Omega \nabla V_k \cdot \nabla \eta + \int_\Omega (\widehat{V}_k \cdot \nabla \widehat{V}_k)\cdot \eta \\
&\quad = \frac{1}{J_k^2}\int_\Omega V_k \cdot \eta + \frac{1}{J_k^2}\int_\Omega x \cdot \nabla V_k \cdot \eta - \frac{1}{J_k^2}\int_\Omega (U_0 \cdot \nabla U_0)\cdot \eta \\
&\qquad - \frac{1}{J_k^2}\int_\Omega U_0 \cdot \nabla V_k \cdot \eta - \frac{1}{J_k^2}\int_\Omega V_k \cdot \nabla U_0 \cdot \eta \\
&\qquad + \frac{L_k}{J_k^2}\int_\Omega \widehat{\Psi}_k \rho \nabla(|\cdot|^{-1})\cdot \eta + \frac{1}{J_k^2}\int_\Omega \Theta_0 \rho \nabla(|\cdot|^{-1})\cdot\eta + \frac{1}{J_k^2}\langle F, \eta\rangle.
\end{aligned}$$

Taking $k \to +\infty$, recalling that

- $J_k \to +\infty$, $L_k/J_k \to 0$,
- $\inf \lambda_k > 0$
- $\widehat{V}_k$ and $\widehat{\Psi}_k$ bounded in $H^1_0(\Omega)$,

we find in the limit

$$\boxed{\int_\Omega (\widetilde{V}\cdot\nabla\widetilde{V})\cdot\eta = 0} \qquad \text{for all } \eta \in C^\infty_{0,\sigma}(\mathbb{R}^3)^3. \tag{6}$$

The boxed identities in the two previous slides give a contradiction.

Indeed, by the latter identity, $\widetilde{V} \in H_0^1(\Omega)^3$ is a stationary solution of the Euler equations: there exists $\widetilde{P} \in L^3(\Omega)$, such that $\|\nabla \widetilde{P}\|_{L^{3/2}(\Omega)} < \infty$, satisfying

$$\begin{cases} \widetilde{V} \cdot \nabla \widetilde{V} = -\nabla \widetilde{P} & \text{in } \Omega \\ \nabla \cdot \widetilde{V} = 0 & \text{in } \Omega \\ \widetilde{V} = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\widetilde{P}(x) \equiv 0$ a.e. on $\partial\Omega$, with respect to the two-dimensional Hausdorff measure [Kapitanskiĭ and Piletskas, 1983]. Then,

$$\int_\Omega (\widetilde{V} \cdot \nabla \widetilde{V}) \cdot U_0 = -\int_\Omega \nabla \widetilde{P} \cdot U_0 = -\int_\Omega \nabla \cdot (\widetilde{P} U_0) = 0.$$

Multiplying by $\lambda_0$,

$$0 = \lambda_0 \int_\Omega (\widetilde{V} \cdot \nabla \widetilde{V}) \cdot U_0 = \boxed{-\lambda_0 \int_\Omega (\widetilde{V} \cdot \nabla U_0) \cdot \widetilde{V} = 1 + \frac{\lambda_0}{2} \int_\Omega |\widetilde{V}|^2}$$

From the last equality we get a contradiction.

Summarizing, the following **a priori estimate** holds for solutions $(V,\Psi)\in\mathbf{H}(\Omega)$ to problem ($\lambda$-PS).

*For some constant* $C_0=C_0(\Omega,F,\rho,U_0,\Theta_0)$, *independent on* $\lambda\in[0,1]$

$$\int_\Omega\Bigl(|V|^2+\Psi^2+|\nabla V|^2+|\nabla\Psi|^2\Bigr)\le C_0.$$

## Existence of solutions to the perturbed elliptic system in bounded domains

### Proposition ($\exists$ in $\Omega$)

*Let $\Omega$ be a bounded domain with a smooth boundary and $\rho \in C_b(\mathbb{R}^3)$, such that $0 \notin \mathrm{supp}(\rho)$. Assume that $F \in (H^{-1}(\Omega))^3$. Then the system on $\Omega$*

$$\begin{cases} -\Delta V + \nabla P - F = V + x \cdot \nabla V - (V + U_0) \cdot \nabla (V + U_0) + (\Psi + \Theta_0)\, \rho \nabla (|\cdot|^{-1}) \\ \qquad -\Delta \Psi = \Psi + x \cdot \nabla \Psi - \nabla \cdot \big( (\Psi + \Theta_0)(V + U_0) \big) \\ \qquad \nabla \cdot V = 0, \\ \qquad V = \Psi = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

*has a solution $(V_\rho, \Psi_\rho) \in \mathbf{H}(\Omega)$.*

**Sketch of proof.** Let $G$ is the map

$$G(V,\Psi) := L(V,\Psi) + N(V,\Psi).$$

where the linear map $L$ and the nonlinear map $N$ are given by

$$L(V,\Psi) := \Big( V + x\cdot\nabla V - U_0\cdot\nabla V - V\cdot\nabla U_0 + \Psi\rho\nabla(|\cdot|^{-1}), \\ \Psi + x\cdot\nabla\Psi - \nabla\cdot\big(\Psi U_0 + \Theta_0 V\big)\Big)$$

$$N(V,\Psi) := \Big( -U_0\cdot\nabla U_0 - V\cdot\nabla V + \Theta_0\rho\nabla(|\cdot|^{-1}), -\nabla\cdot(\Psi V + \Theta_0 U_0)\Big).$$

The following Lemma is easily checked:

**Lemma**

*Let $\Omega$ be a bounded domain with a smooth boundary and $\rho \in C_b(\mathbb{R}^3)$, such that $0 \notin \mathrm{supp}(\rho)$. The nonlinear map $G$ is continuous as a map $G\colon \mathbf{H}(\Omega) \to \mathbf{L}^{3/2}(\Omega)$ and is compact as a map $G\colon \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)'$.*

The weak formulation of our system reads

$$(V,\Psi) = T(G(V,\Psi)) + T((F,0)), \tag{7}$$

where $T\colon \mathbf{H}(\Omega)' \to \mathbf{H}(\Omega)$ is the isomorphism given by the Riesz representation theorem for Hilbert spaces, when $\mathbf{H}(\Omega)$ is endowed with the scalar product

$$((V,\Psi),(V',\Psi')) \mapsto \int_\Omega \nabla V \cdot \nabla V' + \int_\Omega \nabla \Psi \cdot \nabla \Psi'.$$

1. By the Lemma, the nonlinear map

$$(V,\Psi) \mapsto T\circ G(V,\Psi) + T((F,0)) \quad \text{is compact on } \mathbf{H}(\Omega).$$

2. By our first Proposition, if $\lambda \in [0,1]$ and $(V,\Psi) \in \mathbf{H}(\Omega)$ verifies

$$(V,\Psi) = \lambda\big[(T\circ G)(V,\Psi) + T((F,0))\big],$$

   then,

$$\|(V,\Psi)\|_{\mathbf{H}(\Omega)} \le C_0 \qquad (C_0 \text{ independent on } \lambda).$$

The Schaeffer fixed-point theorem implies that the map in the first item has a fixed point $(V_\rho,\Psi_\rho) \in \mathbf{H}(\Omega)$, such that $\|(V_\rho,\Psi_\rho)\|_{\mathrm{H}(\Omega)} \le C_0$, which is a solution of (7)

1. Self-similar solutions of the Navier–Stokes equations. A quick review

2. The Boussinesq system
   - Setting of the problem and statement of the Theorem
   - Analysis of the perturbed system in a bounded domain
   - Existence of solutions of the perturbed elliptic system in the whole space

## Existence of solutions of the perturbed elliptic system in the whole space

Let $k \in \mathbb{N}$, $k \geq 1$. We choose a cut-off function $\rho \in C_b(\mathbb{R}^3)$ such that,

$$\rho(x) = 0 \text{ if } |x| \leq 1/2 \quad \text{and} \quad \rho(x) = 1 \text{ if } |x| \geq 1.$$

Then we set, for $x \in \mathbb{R}^3$,

$$\rho_k(x) := \rho(kx),$$

so that $\rho_k \to 1$ a.e. in $\mathbb{R}^3$ as $k \to +\infty$.

We denote by $(\mathrm{PS})_k$ the perturbated system considered in the previous proposition, with $\rho = \rho_k$ and $\Omega = B_k$ (the open ball centered at the origin of radius $k$).

**Proposition (a priori estimate independent on $k$)**

*Let $F \in H^{-1}(\mathbb{R}^3)$. Let $(V_k, \Psi_k) \in \mathbf{H}(B_k)$ be a solution of problem $(PS)_k$. Then there exists a constant $C_1 = C_1(F, U_0, \Theta_0) > 0$, independent on $k$, such that*

$$\int_{B_k} \left( |V_k|^2 + \Psi_k^2 + |\nabla V_k|^2 + |\nabla \Psi_k|^2 \right) \leq C_1. \tag{8}$$

**Sketch of the proof.** First of all, by estimate (3), in the case $\Omega = B_k$ and $\lambda = 1$, we have

$$\int_{B_k} |\nabla \Psi_k|^2 + \int_{B_k} \Psi_k^2 \leq 2\Big(\|\Theta_0\|^2_{L^\infty(\mathbb{R}^3)} \int_{B_k} |V_k|^2 + \int_{\mathbb{R}^3} |\Theta_0 U_0|^2\Big). \qquad (9)$$

Hence, it is sufficient to prove that

$$\int_{B_k} \big(\frac{1}{2}|V_k|^2 + |\nabla V_k|^2\big) \leq C_1. \qquad (10)$$

With a slight change of notations we now set

$$J_k := \Big(\int_{B_k} \big(\frac{1}{2}|V_k|^2 + |\nabla V_k|^2\big)\Big)^{1/2}, \quad \text{and} \quad L_k := \Big(\int_{B_k} \big(\frac{1}{2}\Psi_k^2 + |\nabla \Psi_k|^2\big)\Big)^{1/2}.$$

Let us assume, by contradiction, that (10) does not hold. Thus, there exists a subsequence of solutions $(V_k, \Psi_k) \in \mathbf{H}(B_k)$ of problem $(\mathrm{PS})_k$ such that

$$J_k \to +\infty.$$

Let

$$\widehat{V}_k := \frac{V_k}{J_k} \quad \text{and} \quad \widehat{\Psi}_k := \frac{\Psi_k}{L_k}.$$

The uniform-in-$k$ boundedness of the sequence $(\widehat{V}_k, \widehat{\Psi}_k)$ in $\mathbf{H}(B_k)$ (and of the sequence obtained extending $(\widehat{V}_k, \widehat{\Psi}_k)$ to the whole $\mathbb{R}^3$), implies that there exists $(\widetilde{V}, \widetilde{\Psi}) \in \mathbf{H}(\mathbb{R}^3)$, such that, after extraction

$$(\widehat{V}_k, \widehat{\Psi}_k) \to (\widetilde{V}, \widetilde{\Psi})$$

weakly in $\mathbf{H}(\mathbb{R}^3)$ and strongly in $\mathbf{L}^p_{\mathrm{loc}}(\mathbb{R}^3)$, for $2 \le p < 6$.

**Step 1. Excluding that $\limsup_{k\to+\infty} J_k/L_k < \infty$.**

Indeed,

$$\limsup_{k\to+\infty} J_k/L_k < \infty \;\Rightarrow \ldots (\text{dividing by } L_k^2) \ldots$$

$$\ldots \Rightarrow \begin{cases} \boxed{1+\gamma\int_{\mathbb{R}^3} \nabla\cdot(\Theta_0 \widetilde{V})\widetilde{\Psi} = 0} & \text{(from the energy identity for } \Psi_k) \\ \boxed{\int_{\mathbb{R}^3} \nabla\cdot(\widetilde{\Psi}\widetilde{V})\chi = 0} \quad \forall\, \chi\in\mathcal{D}(\mathbb{R}^3) & \text{(from the equation of } \Psi_k \text{ in weak form)} \end{cases}$$

These two identities are not compatible.

**Step 2. Excluding that $\limsup_{k\to+\infty} J_k/L_k = \infty$.**

Indeed,

$$\limsup_{k\to+\infty} J_k/L_k = \infty \ \Rightarrow \ldots (\text{dividing by } J_k^2) \ldots$$

$$\ldots \Rightarrow \boxed{\int_{\mathbb{R}^3} (\widetilde{V}\cdot\nabla U_0)\cdot\widetilde{V} = -1}.$$

(from the energy identity for $V_k$)

$$\boxed{\int_{\mathbb{R}^3} (\widetilde{V}\cdot\nabla\widetilde{V})\cdot\eta = 0} \quad \forall\,\eta\in\mathcal{C}^\infty_{0,\sigma}(\mathbb{R}^3)$$

(from the equation of $V_k$ in weak form)

These two identities finally lead to a contradiction.

**Proposition ($\exists$ in $\mathbb{R}^3$)**

*Let $F \in H^{-1}(\mathbb{R}^3)^3$. Then the elliptic system* (PS) *possess at least one solution* $(V, \Psi) \in \mathrm{H}(\mathbb{R}^3)$.

**Sketch of the proof.**

- By Proposition 2, with $\Omega = B_k$ and $\rho = \rho_k$ $(k = 1, 2, \ldots)$, we obtain the existence of a solution $(V_k, \Psi_k) \in \mathbf{H}(B_k)$.
- By Proposition 3, such this sequence of solutions is bounded in the $\mathbf{H}(B_k)$-norm by a constant independent on $k$. Then $\exists (V, \Psi) \in \mathbf{H}(\mathbb{R}^3)$ and a subsequence, such that
$$(V_k, \Psi_k) \to (V, \Psi) \text{ weakly in } \mathbf{H}(\Omega),$$
for any bounded domain $\Omega \subset \mathbb{R}^3$.
- The passage to the limit to see that $(V, \Psi)$ is a weak solution of the elliptic problem (PS) is standard.

## Conclusion

Let $(V, \Psi)$ as above. Recall that

$$U_0 = e^{\Delta/2} u_0, \qquad \text{and} \qquad \Theta_0 = e^{\Delta/2}\theta_0.$$

Let us define

$$u(x,t) := \frac{1}{\sqrt{2t}}(U_0 + V)\Big(\frac{x}{\sqrt{2t}}\Big) \quad \text{and} \quad \theta(x,t) := \frac{1}{\sqrt{2t}}(\Theta_0 + \Psi)\Big(\frac{x}{\sqrt{2t}}\Big).$$

Then $(u, \theta)$ is a self-similar solution of the Boussinesq system.

Moreover, we have $(U_0, \Theta_0) \in \mathbf{L}^{3,\infty}(\mathbb{R}^3)$ and $(V, \Psi) \in \mathbf{H}(\mathbb{R}^3) \subset \mathbf{L}^{3,\infty}(\mathbb{R}^3)$. Then, from the scaling properties

- $(u, \theta) \in L^\infty(\mathbb{R}^+, \mathbf{L}^{3,\infty}(\mathbb{R}^3))$.
  In fact, $(u, \theta) \in C_w([0, \infty), \mathbf{L}^{3,\infty}(\mathbb{R}^3))$.
- for some constants $c, c' \geq 0$ and all $t > 0$,

$$\|u(t) - e^{t\Delta}u_0\|_2 + \|\theta(t) - e^{t\Delta}\theta_0\|_2 = ct^{1/4},$$
$$\|\nabla u(t) - \nabla e^{t\Delta}u_0\|_2 + \|\nabla\theta(t) - \nabla e^{t\Delta}\theta_0\|_2 = c't^{-1/4}.$$

Our theorem is established. □

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$$\|u(t) - e^{t\Delta}u_0\|_2 + \|\theta(t) - e^{t\Delta}\theta_0\|_2 = ct^{1/4},$$
$$\|\nabla u(t) - \nabla e^{t\Delta}u_0\|_2 + \|\nabla\theta(t) - \nabla e^{t\Delta}\theta_0\|_2 = c't^{-1/4}.$$

Our theorem is established. $\square$

**THANKS !**