Comportement en temps long du système de Stokes-transport

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Outline

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Context and motivation

Goal: Study long-time behavior of stratified fluids in the presence of boundaries.

Motivation: oceanography: on large scales, fluid is described by Boussinesq/primitive equations.

Stratification plays an important role in the stability, but interaction with boundary still poorly understood.

Today's talk: step towards understanding interplay between stratification/solid boundaries, with simplified model.

The Stokes-transport system

Variables: density ρ, velocity $\mathbf{u} : [0,\infty) \times \Omega \to \mathbf{R}^2$:

$$(ST) \qquad \begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0 \\ -\Delta \mathbf{u} + \nabla p = -\rho \mathbf{e}_z \\ \operatorname{div} \mathbf{u} = 0 \\ \rho|_{t=0} = \rho_0 \end{cases}$$

Domain and BC: $\Omega = \mathbf{T} \times (0,1)$, no-slip BC for $\mathbf{u}$.
Derivation of the system:

- Boussinesq $\to$ neglect advection (large Prandtl number) [Grayer II, 2024; Lazar, Xue, Zhang, 2024];
- Homogenization of sedimenting particle systems [Höfer, 2018; Mecherbet, 2021];

Remark: Steady states: $\rho = \rho(z)$, $\mathbf{u} = 0$.
GWP: [Leblond, 2022], with $\rho_0 \in L^\infty$.

Related works

- **IPM system:**
  Replace Stokes with Darcy's law $u + \nabla p = -\rho \mathbf{e}_z$:
  - LWP in Sobolev spaces [Córdoba, Gancedo, Orive, 2007]: much harder than for (ST);
  - Long time behavior around stratified eq.: [Elgindi, 2017] (no boundary), [Castro, Córdoba, Lear, 2019; Park 2024].
- **Boussinesq system:** with damping [Castro, Córdoba, Lear, 2019], with viscosity [Park 2024].
- **Interface problem:** LWP $\rho_0 = \mathbf{1}_{z<\eta_0(x)}$, long-time stability of $\mathbf{1}_{z<z_0}$ [Gancedo, Granero-Belinchón, Salguero, 2022].
- **Low regularity instability:** growth of Sobolev norms for small H^{2-} perturbations of stratified data [Kiselev & Yao, 2021] (for IPM, extended to (ST) by [Leblond, 2024]).

About long time stability

Look at solutions around profile $\rho_s(z) = 1 - z$.
Remark: decay of energy $\to$ expect stability.
Other works in similar setting: IPM with $\rho_0 - \rho_s \in H_0^k$
[Castro, Córdoba, Lear, '19; Park '24]

$$\|\rho(t) - \rho_\infty(z)\|_{L^2} \lesssim \frac{\|\rho_0 - \rho_s\|_{H^k}}{t^{k/2}}.$$

$\to$ Arbitrarily high decay for smooth enough perturbations.
Remark: same result/techniques probably work for Stokes or Boussinesq with slip BC. Decay rate $t^{-k/4}$.
Question: same result with no-slip BC?
Answer: No!

Main results - 1 - Stability

Goal: Long time stability of profile $\rho_s = \rho_s(z)$ s.t. $\partial_z \rho_s < 0$ (e.g. $\rho_s(z) = 1 - z$).
Assumptions on initial data: $\|\rho_0 - \rho_s\|_{H^k} \ll 1$ for some sufficiently large k, $\partial_z^j(\rho_0 - \rho_s)|_{\partial\Omega} = 0$, $j = 0, 1$.

Theorem 1: [D., Guillod, Leblond, 2023]
Let $\rho^* = \rho^*(z)$ be the vertical rearrangement of ρ_0.
Then for all $t \geq 0$,

$$\|\rho(t) - \rho^*\|_{L^2} \lesssim \frac{\|\rho_0 - \rho_s\|_{H^k}}{1 + t},$$
$$\|\rho(t) - \rho^*\|_{H^4} \lesssim \|\rho_0 - \rho_s\|_{H^k}.$$

"Scaling": 1 z-derivative $\leftrightarrow$ loss of $t^{1/4}$.

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Main results - 2 - Boundary layers hinder decay

Recall:

- $\rho(t) - \rho^* = O(t^{-1})$ in L^2, $O(1)$ in H^4.
- 1 z-derivative $\leftrightarrow$ loss of $t^{1/4}$.

Theorem 2: [D., Guillod, Leblond, 2023]

$$\rho(t) - \rho^* = \rho^{\mathrm{BL}} + \rho^{\mathrm{int}}$$

where

$$\rho^{\mathrm{BL}} = \underbrace{t^{-1}\Theta^0(x, t^{1/4}z) + t^{-1}\Theta^1(x, t^{1/4}(1-z))}_{\text{size } t^{-9/8} \text{ in } L^2} + \text{l.o.t.}$$

and $\Theta^j(x, Z)$ decay exp. as $Z \to \infty$, and

$$\|\rho^{\mathrm{int}}\|_{L^2} = O(t^{-2}), \quad \|\rho^{\mathrm{int}}\|_{H^8} = O(1).$$

Remark: Energy concentrated in BL of size $t^{-1/4}$.
Boundaries slow down decay: $\|\rho - \rho^*\|_{H^k} \sim t^{-1+\frac{k}{4}-\frac{1}{8}}$.

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Preliminary decomposition

Ideas:

- Separate the averaged/oscillating parts (cf. [Elgindi, 2017]):

$$\theta := \rho - \rho_s = \bar{\theta} + \theta' = \int_{\mathbf{T}} \theta + \theta';$$

- Introduce stream function ψ s.t. $u = \nabla^\perp \psi$, which solves

$$\Delta^2 \psi = \partial_x \theta', \quad \psi|_{\partial\Omega} = \partial_n \psi|_{\partial\Omega} = 0.$$

Ansatz: $\theta'(t) \to 0$, $\bar{\theta}(t,z) \to \theta_\infty(z)$ as $t \to \infty$; $\|\theta\| \lesssim \|\theta_0\| \ll 1$.
Eq. for θ' and $\bar{\theta}$ when $\rho_s = 1 - z$:

$$\partial_t \theta' = \partial_x \psi \underbrace{-\partial_z \bar{\theta} \partial_x \psi - (\nabla^\perp \psi \cdot \nabla \theta')'}_{\text{negligible}},$$

$$\partial_t \bar{\theta} = -\overline{\nabla^\perp \psi \cdot \nabla \theta'} = O(t^{-1-\delta}).$$

Sketch of proof of stability

Structure of eq. on θ':

$$\partial_t \theta' = (1 - \underbrace{\partial_z \bar{\theta}}_{\ll 1}) \partial_x^2 \Delta^{-2} \theta' + Q(\theta', \theta')$$

1. **Analysis of the linear semi-group** $\exp(t\partial_x^2 \Delta^{-2})$**:**
   - Algebraic decay, but no regularizing effect;
   - Trade regularity/decay;
2. **Bootstrap argument:**
   - θ' enjoys decay predicted by linear analysis;
   - Quadratic term $Q(\theta', \theta')$ remains negligible;
   - $\bar{\theta}$ remains small and converges as $t \to \infty$;
3. **Identification of the limit:**
   - $\rho(t) \to \rho^*(z)$ as $t \to \infty$;
   - Level sets of $\rho(t)$ have constant measure.

   Consequence: $\rho^* =$ vertical rearrangement of ρ_0.

Spectral analysis of linearized operator

Lemma [Leblond, 2023] $\exists$ *orthogonal family* $(\theta_{k,n})_{k\in\mathbf{Z},n\in\mathbf{N}}$ *of eigenfunctions*

$$\Delta^2\theta_{k,n} = \lambda_{k,n}\theta_{k,n}, \quad \theta_{k,n} = \partial_n\theta_{k,n} = 0 \text{ on } \partial\Omega,$$

and

$$\lambda_{k,n} \simeq (k^2+n^2)^2.$$

Consequence:

$$\exp(\partial_x^2\Delta^{-2}t)\theta_0' = \sum_{k\neq 0,n} \exp\left(-\frac{k^2}{\lambda_{k,n}}t\right)\langle\theta_0', \theta_{k,n}\rangle\theta_{k,n}.$$

Quantitative decay estimate:

$$\left\|\exp(\partial_x^2\Delta^{-2}t)\theta_0'\right\|_{L^2} \lesssim t^{-1}\|\Delta^2\partial_x^{-2}\theta_0'\|_{L^2}.$$

- Trade regularity for decay;
- Gain $t^{1/4}$ for each (z)-derivative;
- IPM: replace Δ^2 by Δ.

Derivation of a uniform H^4 bound

Key step: prove that $\sup_t \|\theta'\|_{H^4} < \infty$.
(Then linear analysis $\Rightarrow$ decay of L^2 norm.)
Back to eq.:

$$\partial_t \theta' = (1 - \underbrace{\partial_z \bar{\theta}}_{\ll 1}) \partial_x^2 \Delta^{-2} \theta' + Q(\theta', \theta')$$

Apply Δ^2:

$$\partial_t \Delta^2 \theta' = (1 - \partial_z \bar{\theta}) \partial_x^2 \theta' + \text{l.o.t.}$$

Assumptions on θ_0: $\theta' = \partial_n \theta' = 0$ on $\partial\Omega \to$ IBP.
H^4 estimate:

$$\frac{d}{dt} \|\Delta^2 \theta'\|_{L^2}^2 + \|\partial_x \Delta \theta'\|_{L^2}^2 \leq \text{l.o.t.}$$

Higher decay estimates?

Key information to prove uniform H^4 bound:

$$\theta'(t) = \partial_n \theta'(t) = 0 \quad \text{on } \partial\Omega.$$

$\rightarrow$ Preserved by the evolution.
What about $\Delta^2\theta'|_{\partial\Omega}$?

$$\partial_t \Delta^2 \theta'|_{z=0} = -6\partial_z^3\bar{\theta}\partial_x\partial_z^2\psi|_{z=0} + \text{l.o.t.} \neq 0.$$

Consequence: no uniform H^8 bound & no decay of H^4 norm.
⚠Different from IPM ! [2019, Castro, Córdoba, Lear]
Remark: if no-slip condition is replaced with perfect slip

$$\mathbf{u} \cdot n = \partial_n \mathbf{u}_\tau = 0 \quad \text{on } \Omega,$$

then argument can be repeated: decay estimates at any order.

Summary

- Proof of stability thanks to coercivity of linearized operator + bootstrap argument;
- Argument cannot be exported to higher regularity: important difference with previous works on Boussinesq/IPM;
- Issue comes from boundary terms. In the case without boundary, for initial data in H^{4n+}, one can
  1. Derive uniform estimates on $\Delta^{2n}\theta'$;
  2. Deduce that $\|\theta'\|_{H^{4k}} = O(t^{k-n})$ for $0 \leq k \leq n$.

Question: actual or technical limitation?

Outline

Setting of the problem

Reminder:

$$\partial_t \Delta^2 \theta'|_{\partial\Omega} = O(t^{-1-\delta}) \neq 0.$$

Consequence: as $t \to \infty$, $\exists \gamma^0, \gamma^1$,

$$\Delta^2 \theta'|_{\partial\Omega} \to \gamma^0, \quad \partial_n \Delta^2 \theta'|_{\partial\Omega} \to \gamma^1.$$

Remak: γ^0, γ^1 depend on whole nonlinear evolution.
Question: what is the influence of γ^0, γ^1 on the dynamic as $t \to \infty$?

A linear toy model for high order derivatives

Good toy model for $\Delta^2\theta'$:

$$\partial_t \eta = \partial_x^2 \Delta^{-2}\eta, \quad \eta_{||t=0} = \eta_0,$$

and η_0 is such that

$$\eta_0|_{\partial\Omega} \neq 0, \quad \partial_n \eta_0|_{\partial\Omega} \neq 0.$$

Observations:

- Spectral analysis + Lebesgue theorem $\Rightarrow \eta(t) \to 0$ in L^2;
- BUT $\eta|_{\partial\Omega}$ and $\partial_n \eta|_{\partial\Omega}$ remain constant!

Idea: solution concentrates close to boundaries.
Boundary layer size? Recall $\partial_z \leftrightarrow t^{1/4}$.
$\to$ BL of width $t^{-1/4}$ (self-similar behavior).

Boundary layer formation in the linear TM

$$\partial_t \eta = \partial_x^2 \Delta^{-2} \eta, \quad \eta\big|_{|t=0} = \eta_0.$$

Ansatz: close to $z = 0$,

$$\eta \simeq H^0(x, t^{1/4} z), \quad \psi = \Delta^{-2} \partial_x \eta \simeq t^{-1} \Psi^0(x, t^{1/4} z).$$

Plug into eq.: setting $Z = t^{1/4} z$,

$$\frac{1}{4} Z \partial_Z H^0 = \partial_x \Psi^0, \quad \partial_Z^4 \Psi^0 = \partial_x H^0.$$

Closed eq. for Ψ^0:

$$Z \partial_Z^5 \Psi^0 = 4 \partial_x^2 \Psi^0,$$
$$\Psi^0_{|Z=0} = \partial_Z \Psi^0_{|Z=0} = 0, \; \partial_Z^4 \Psi^0_{|Z=0} = \partial_x \eta_{0|z=0}.$$

Long time behavior of the linear TM

Eq on the boundary layer profiles:

$$\text{(BL)} \quad Z\partial_Z^5\Psi^0 = 4\partial_x^2\Psi^0,\ \Psi^0_{|Z=0} = \partial_Z\Psi^0_{|Z=0} = 0,\ \partial_Z^4\Psi^0_{|Z=0} = \partial_x\eta_{0|z=0}.$$

Lemma: $\exists!$ *sol. of* (BL) *such that*

$$|\Psi^0(x,Z)| \lesssim \exp(-cZ^{4/5}).$$

Define $\eta^{\mathrm{BL}} = H^0(x, t^{1/4}z) + H^1(x, t^{1/4}(1-z)) +$ l.o.t.
Then $\eta - \eta^{\mathrm{BL}}$...

- is an approx. sol. of $\partial_t\eta = \partial_x^2\Delta^{-2}\eta$;
- vanishes on $\partial\Omega$ (+ normal derivative).

Conclusion: (cf. previous section:)

$$\|\eta - \eta^{\mathrm{BL}}\|_{L^2} = O(t^{-1}).$$

Remark: $\|\eta^{\mathrm{BL}}\|_{L^2} \simeq t^{-1/8}$:
All the energy is focused in the BL.

Back to the Stokes-transport system...

Intuition: $\Delta^2\theta' \sim H^0(x, t^{1/4}z)$ for $t \gg 1$, $0 < z \ll 1$.
New Ansatz:

$$\theta' = t^{-1}\Theta^0(x, t^{1/4}z) + t^{-1}\Theta^1(x, t^{1/4}(1-z)) + \text{l.o.t.} + \theta^{\text{int}}.$$

Now, by definition of Θ^0, Θ^1,

$$\Delta^2\theta^{\text{int}}_{|\partial\Omega} = \partial_n\Delta^2\theta^{\text{int}}_{|\partial\Omega} = 0.$$

Apply first stability result to θ^{int}:

$$\|\Delta^4\theta^{\text{int}}\|_{L^2} = O(1),\ \|\Delta^2\theta^{\text{int}}\|_{L^2} = O(t^{-1}),\ \|\theta^{\text{int}}\|_{L^2} = O(t^{-2})$$

(To be compared with $\|\Delta^2\theta'\|_{L^2} = O(1)$, $\|\theta'\|_{L^2} = O(t^{-1})$.)

Remarks on the proof

- Need for good enough approximation
  $\rightarrow$ Construction of several correctors.
- Structure of higher order BL terms Θ_j, $j \geq 1$

  Linear op. (Θ_j)=quadratic terms depending on $\Theta_k, k < j$.

  $\simeq$ Weakly nonlinear construction.
- In the linear setting, expansion can be pushed at arbitrary order.
  Nonlinear case: probably so, but high technical cost!
- Intricate bootstrap argument on θ^{int}.

Outline

1. Arbitrary initial data

Question: What happens when $\theta_0|_{\partial\Omega} \neq 0$?
Then $\theta(t)|_{\partial\Omega} = \theta_0|_{\partial\Omega} \neq 0$ $\forall t \geq 0$.
New Ansatz: $\theta(t, x, z) \simeq \Theta^0(x, t^\alpha z)$ for $0 < z \ll 1$, for $\alpha > 0$.
Plug into eq. [...] $\to$ $\alpha = 1/3$ and

$$\frac{1}{3} Z \partial_Z \Theta^0 + \{\Psi^0, \Theta^0\} = 0, \quad \partial_Z^4 \Psi^0 = \partial_x \Theta^0.$$

Remarks:

- Change of BL size ($t^{-1/3}$ vs. $t^{-1/4}$);
- Nonlinear at main order;
- Well-posedness of BL eq. is unclear! (Loss of derivatives?)

Questions: WP of the BL eq. ? Justification of the Ansatz?

2. Boussinesq system with no-slip BC

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \Delta\mathbf{u} = -\rho\mathbf{e}_z,$$
$$\partial_t\rho + \mathrm{div}(\mathbf{u}\rho) = 0.$$

Observation: With the same Ansatz as before, i.e.

$$\mathbf{u} = \nabla^\perp\psi \simeq t^{-2}\nabla^\perp\Psi^0(x, t^{1/4}z)$$

for $t \gg 1$, $0 < z \lesssim t^{-1/4}$,

$$\partial_t u_1 + (\mathbf{u}\cdot\nabla)u_1 = O(t^{-1-2+\frac{1}{4}} + t^{-\frac{7}{4}-\frac{7}{4}}) = O(t^{-11/4}),$$
$$\Delta u_1 = O(t^{-2+\frac{1}{4}+1}) = O(t^{-3/4}) \gg t^{-11/4}.$$

$\rightarrow$ Advection is negligible.

Question: Long-time behavior of Boussinesq with boundaries?

3. IPM model

Reminder: [Castro, Cordóba, Lear, 2019; Park, 2024]

- If $\partial_z^{2k}\theta_{0|\partial\Omega} = 0$ for $0 \leq k \leq k^*$: preserved by the evolution;
- If $\theta_0 \in H_0^k(\Omega)$, $k \geq 3$, $\|\theta_0\|_{H^k} \ll 1$, then $\|\rho - \rho^*\|_{L^2} \lesssim \|\theta_0\|_{H^k} t^{-k/2}$.

Question: What about the case $\theta^0_{|\partial\Omega} \neq 0$? Or $\partial_z^2\theta^0_{|\partial\Omega} \neq 0$?

Conjecture:

- Same type of linear BL as for Stokes-transport when $\theta^0_{|\partial\Omega} = 0$ (size $t^{-1/2}$);
- Nonlinear BL when $\theta^0_{|\partial\Omega} \neq 0$.

4. Infinite channel

Domain: $\Omega = \mathbf{R} \times (0,1)$, initial data $\rho_0 \in L^\infty \cap H^s_{\mathrm{uloc}}$.
Global well-posedness: [Leblond, 2022].
Questions: stability of the profile ρ_s? Boundary layer formation?
Issues:

- Control of small horizontal frequencies (lack of spectral gap).
- What about the decomposition $\theta = \bar{\theta} + \theta'$? Notion of average?

Conclusion

- Long time behavior of Stokes-transport system in the presence of boundaries;
- Stability of stratified profiles for smooth initial data s.t. $\partial_z^k(\rho^0 - \rho_s)_{|d\Omega} = 0,\ k = 0, 1, 2$;
- Boundaries slow down the convergence;
- Energy gets trapped in boundary layer (size $t^{-1/4}$);
- Could be extended to other fluid models (IPM, Boussinesq).
- One last perspective: non flat boundary???

Thank you for your attention!

Joyeux anniversaire!