On the large time asymptotics of global solutions to the Vlasov-Navier-Stokes equations in the whole space

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The Vlasov-Navier-Stokes equations (VNS)

$$(VNS) \qquad \begin{cases} f_t + v\cdot\nabla_x f + \mathrm{div}_v\big(f\,(u-v)\big) = 0 & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3, \\ u_t + u\cdot\nabla_x u - \Delta_x u + \nabla_x P = \displaystyle\int_{\mathbb{R}^3} f\,(v-u)\,dv & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3, \\ \mathrm{div}_x u = 0 & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3. \end{cases}$$

Distribution of particles : $f = f(t,x,v) \in \mathbb{R}_+$.

Fluid velocity field : $u = u(t,x) \in \mathbb{R}^3$.

Pressure : $P = P(t,x) \in \mathbb{R}$.

Kinetic variable : $v \in \mathbb{R}^3$.

Space variable : $x \in \mathbb{R}^3$.

Time variable : $t \in \mathbb{R}_+$.

Brinkman force : $\displaystyle\int_{\mathbb{R}^3} f\,(v-u)\,dv.$

Basic relations

- **Energy balance :**

$$\frac{d}{dt}E_0 + D_0 = 0 \quad \text{with} \quad E_0 := \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^3_x)} + \frac{1}{2}\||v|^2 f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}$$
$$\text{and} \quad D_0 := \|\nabla u\|^2_{L^2(\mathbb{R}^3_x)} + \||u-v|^2 f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}.$$

- **Continuity equation :** the density $\rho(t,x) := \int f(t,x,v)\,dv$ and momentum $j(t,x) := \int v f(t,x,v)\,dv$ satisfy $\partial_t\rho + \mathrm{div}\, j = 0.$

- **Mass conservation :** $\|\rho(t)\|_{L^1} = M_0 := \|\rho_0\|_{L^1}.$

- **Formula for the distribution of particles :**

$$f(t,x,v) = e^{3t} f_0(X(0;t,x,v);V(0;t,x,v)) \quad \text{with} \quad (X,V) \text{ flow of } (v, u-v).$$

Weak solutions theory

Theorem (Boudin, Desvillettes, Grandmont and Moussa, 2009). *Let*

$$f_0 \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \quad \textit{with} \quad |v|^2 f_0 \in L^1(\mathbb{T}^3 \times \mathbb{R}^3), \qquad u_0 \in L^2(\mathbb{T}^3) \quad \textit{with} \quad \operatorname{div} u_0 = 0.$$

Then (VNS) admits a global-in-time distributional solution verifying

$$E_0(t) + \int_0^t D_0 \, d\tau \leq E_{0,0} := \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3_x)} + \frac{1}{2}\||v|^2 f_0\|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)}, \quad \textit{a.e.} \ t \in \mathbb{R}_+.$$

• A similar result holds true in $\mathbb{R}^2$, $\mathbb{T}^2$, $\mathbb{R}^3$, and in bounded two or three-dimensional domains.

• In dimension two, weak solutions are unique (Han-Kwan, Miot, Moussa and Moyano, 2020).

Goal of the talk

- Constructing global-in-time unique solutions in the $\mathbb{R}^3$ case under a ‘critical’ smallness condition.

- Specifying the long-time behavior, in particular the rate of convergence to 0 of the energy E_0 and the limit of the distribution of particles f.

State of the art: $u_0 \in H^1(\mathbb{R}^3)$ $+$ some positive Besov regularity $+$ *strong localization of* f_0 $+$ smallness. Global existence and almost optimal decay rate of E_0 are proved, and $f \to \rho \otimes \delta_{v=0}$ (Han Kwan ’22).

Aim: only H^1 regularity ($H^{1/2}$ is enough), proving optimal decay and pointing out that $\rho \otimes \delta_{v=u}$ is a better approximatiom of f.

Theorem (R.D. 2024). *Let* $u_0 \in L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$,

$$f_0 \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) \cap L^\infty(\mathbb{R}^3_v \times \mathbb{R}^3_x) \quad \textit{and} \quad |v|^2 f_0 \in L^1(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x)).$$

There exists a small constant c_0 *depending only on* $\|u_0\|_{L^1(\mathbb{R}^3)}$ *and on norms of* f_0 *such that, if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \iint |v|^2 f_0 \, dv \, dx \le c_0,$$

then (VNS) *admits a unique global solution* (f, u, P) *satisfying* $\forall\, t > 0$

$$E_0(t) + \int_0^t D_0 \, d\tau = E_{0,0} \quad \textit{and} \quad \|f(t)\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} = M_0 := \|f_0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}$$

and such that:

- $u \in \mathscr{C}_b(\mathbb{R}_+; H^1) \cap L^2(\mathbb{R}_+; L^\infty)$, $\nabla u \in L^1(\mathbb{R}_+; L^\infty)$,
- $\nabla P, \nabla^2 u \in L^2(\mathbb{R}_+ \times \mathbb{R}^3)$,
- $f \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^3_v \times \mathbb{R}^3_x))$, $|v|^2 f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x)))$.

Asymptotic behavior

Theorem (R.D. 2024). *The following 'heat-like' time decay estimates hold for* E_0 *and* $E_1 := \|\nabla u\|_{L^2}^2 + \iint f|u-v|^2\,dx\,dv$*:*

$$E_0(t) \le C_0(1+a_0 t)^{-3/2} \quad \textit{and} \quad E_1(t) \le C_1(1+a_1 t)^{-5/2}.$$

Furthermore, there exists j_∞ *in* $L^1(\mathbb{R}^3)$ *such that*

$$\|\rho(t) - \rho_0 + \operatorname{div} j_\infty\|_{\dot W^{-1,1}} \le C_0 t^{-1/4}$$

and we have

$$\|(j-\rho u)(t)\|_{L^1} + W_1(f(t), \rho(t)\otimes\delta_{v=u(t)}) \le C_0 t^{-5/4}$$

where the Wasserstein distance $W_1(\mu,\nu)$ *is given by*

$$W_1(\mu,\nu) := \sup\Big\{\iint \phi\,d\mu(x,v) - \iint \phi\,d\nu(x,v),\ \ \phi\in C^{0,1}(\mathbb{R}^3\times\mathbb{R}^3),\ \ \|\nabla_{x,v}\phi\|_{L^\infty} = 1\Big\}.$$

Proof of the asymptotic behavior

It is based on the fact that $E_1(t) \leq C_1(1 + a_1 t)^{-5/2}$.

By definition of $\rho(t)$ and of the Wasserstein distance W_1 between $f(t)$ and $\rho(t) \otimes \delta_{u(t)}$, and of $\rho(t)$, we have

$$W_1(f(t), \rho(t) \otimes \delta_{u(t)}) = \sup_{\|\nabla_{x,v}\phi\|_{L^\infty}=1} \iint f(t,x,v)\big(\phi(x,v) - \phi(x,u(t,x))\big)\,dv\,dx.$$

For all function ϕ with $\|\nabla_{x,v}\phi\|_{L^\infty} = 1$ we can write:

$$\begin{aligned}\iint f(t,x,v)\big(\phi(x,v) - \phi(x,u(t,x))\big)\,dx\,dv &\leq \iint f(t,x,v)|v - u(t,x)|\,dv\,dx \\ &\leq \Big(\int \rho(t)\,dx \iint f(t)|v-u(t)|^2\,dv\,dx\Big)^{1/2} \\ &\leq \sqrt{M_0 E_1(t)} \leq C_2(1 + a_0 t)^{-5/4}.\end{aligned}$$

Similarly, $\|j(t) - (\rho u)(t)\|_{L^1} \leq \iint f(t)|v-u(t)|\,dv\,dx \leq \sqrt{M_0 E_1(t)} \leq C_2(1 + a_0 t)^{-5/4}$.

The main steps leading to the global existence

1. Basic energy balance $\frac{d}{dt}E_0 + D_0 = 0.$

2. Higher order energy inequality (pointwise control of $E_1 := D_0$ and of $E_2 \approx \|u_t\|^2_{L^2}$). Requires ρ bounded and $\nabla u \in L^1(\mathbb{R}_+; L^\infty)$.

3. Control of $\|\rho\|_{L^\infty}$ assuming that $f_0 \in L^1(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x))$ and that

$$\text{(Lip)} \qquad \int_0^\infty \|\nabla u\|_{L^\infty}\, dt \leq 1/9.$$

This follows from Han-Kwan, Moussa and Moyano's paper.

4. Proving optimal time decay of E_0 and of E_1.

5. Proving (Lip): follows from Steps 2 and 4, and interpolation.

In contrast with NS, our approach requires $u_0 \in L^p$ for $p \simeq 1$.

The basic energy balance

$$\frac{d}{dt}E_0 + D_0 = 0 \quad \text{with} \quad E_0 := \frac{1}{2}\int_{\mathbb{R}^3} |u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f\, dx\, dv$$

$$\text{and} \quad D_0 := \int_{\mathbb{R}^3} |\nabla u|^2\, dx + \int_{\mathbb{R}^3\times\mathbb{R}^3} |v-u|^2 f\, dx\, dv.$$

Control of higher order energy functionals : step 1

Aim: Getting a global control of $E_1 := \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\iint |v-u|^2 f\,dx\,dv.$

- A priori assumption: ρ is bounded and $\nabla u \in L^1(\mathbb{R}_+; L^\infty)$.

Take scalar product of velocity equation with u_t:

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 = -\int (u\cdot\nabla u)\cdot u_t\,dx + \iint u_t\cdot((v-u)f)\,dv\,dx.$$

Basic computations (like IPP) give

$$\iint u_t\cdot((v-u)f)\,dv\,dx = -\frac{1}{2}\frac{d}{dt}\iint |v-u|^2 f\,dv\,dx - \iint f|v-u|^2\,dv\,dx$$
$$+\iint f(v-u)\cdot(u\cdot\nabla u)\,dv\,dx + \iint f(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx.$$

Hence E_1 and $\widetilde{D}_1 := \int |u_t|^2\,dx + \iint f|v-u|^2\,dv\,dx$ satisfy

$$\frac{d}{dt}E_1 + \widetilde{D}_1 = -\int (u\cdot\nabla u)\cdot u_t\,dx + \iint f(v-u)\cdot(u\cdot\nabla u)\,dv\,dx + \iint f(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx.$$

The r.h.s. is bounded by Young, Hölder and Gagliardo-Nirenberg inequalities. Denoting $U := \|\nabla u\|_{L^\infty}$ and $R := \max(1, \|\rho\|_{L^\infty})$, we get

$$\frac{d}{dt}E_1 + \widetilde{D}_1 \leq \varepsilon\|\nabla^2 u\|_{L^2}^2 + CUE_1 + C\varepsilon^{-1}R^2\|\nabla u\|_{L^2}^6 \text{ for all } \varepsilon > 0.$$

To bound $\|\nabla^2 u\|_{L^2}^2$, we use elliptic regularity of the Stokes system:

$$-\Delta u + \nabla P = -u_t - u\cdot\nabla u + \int f(v-u)\,dv, \qquad \operatorname{div} u = 0.$$

Choosing $\varepsilon = R^{-1}$ and setting $D_1 := \frac{1}{2}\widetilde{D}_1 + \frac{1}{24R}\|\nabla^2 u, \nabla P\|_{L^2}^2$, we get

$$\frac{d}{dt}E_1 + D_1 \leq CUE_1 + CR^3 D_0 E_1^2.$$

Gronwall yields all-time control if $E_{1,0} \ll 1$, $R < \infty$ and $U \in L^1(\mathbb{R}_+)$.

Control of a higher order energy functional : step 2

To get a relation interrelating

$$E_2 := D_1 = \|u_t\|^2_{L^2} + \iint f|v-u|^2\,dv\,dx \quad \text{and} \quad D_2 := \|\nabla u_t\|^2_{L^2} + \|\sqrt{\rho}\,u_t\|^2_{L^2},$$

take the scalar product of u_t with

$$u_{tt} + u\cdot\nabla u_t + \nabla P_t - \Delta u_t + \rho u_t + u_t\cdot\nabla u = \int (u-v)(v\cdot\nabla_x f)\,dv + \int (v-u) f\,dv.$$

⚠ For $u_0 \in H^1$, $u_t|_{t=0} \notin L^2$.

Parabolic scaling: $\sqrt{t}\nabla_x$ is of order 0 $\rightsquigarrow$ we look at tE_2. We get:

$$\frac{d}{dt}(tE_2) + tD_2 + \iint f|v-u|^2\,dv\,dx \le 24RD_1 + 2\big(1+\|m_2\|_{L^\infty}\big)tD_1$$
$$+ C\|\rho\|^2_{L^\infty} D_0 (tE_1)^2 + C\|\rho\|_{L^\infty}\sqrt{RD_0D_1}\,tE_1 + \big(2\|\nabla u\|_{L^\infty} + C\|\rho\|_{L^\infty}\sqrt{RD_0D_1}\big)tE_2.$$

Control of the L^∞ bound of the density

We follow Han Kwan-Moussa-Moyano: let $Z=(X,V)$ satisfy the ODE:

$$\begin{cases} \partial_s X(s;t,x,v) = V(s;t,x,v) \\ \partial_s V(s;t,x,v) = u(s;X(s;t,x,v)) - V(s;t,x,v) \\ X(t;t,x,v) = x \quad \text{and} \quad V(t;t,x,v) = v. \end{cases}$$

Since $f_t + v\cdot\nabla_x f - 3f + (u-v)\cdot\nabla_v f = 0$, we have

$$f(t,x,v) = e^{3t} f_0\big(X(0;t,x,v), V(0;t,x,v)\big),$$

and thus

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv = e^{3t}\int_{\mathbb{R}^3} f_0\big(X(0;t,x,v), V(0;t,x,v)\big)\,dv.$$

Change of variable : Z_t is a bi-Lipschitz diffeomorphism on $\mathbb{R}^3\times\mathbb{R}^3$, but what about $\Gamma_{t,x} : v \mapsto V(0;t,x,v)$?

(Lip) implies $\det D_v \Gamma_{t,x} \geq e^{3t}/2$, and thus $\rho(t,x) \leq 2\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}$.

Final a priori estimate

Use the following combination of (E_0, D_0), (E_1, D_1) and (E_2, D_2):

$$\mathscr{E}(t) := 2(2 + \|m_2\|_{L^\infty})(tE_1(t) + 2E_0(t)) + 25RE_1(t) + tE_2(t)$$

and $$\mathscr{D}(t) := 2(1 + \|m_2\|_{L^\infty})D_0(t) + 2tD_1(t) + RD_1(t) + tD_2(t).$$

We get

$$\frac{d}{dt}\mathscr{E} + \mathscr{D} \leq C\big(U\mathscr{E} + \mathscr{D}\mathscr{E} + \|\rho\|_{L^\infty}^2 D_0 \mathscr{E}^2\big) \quad \text{with} \quad U := \|\nabla u\|_{L^\infty}.$$

Use also flow estimates and Gronwall lemma to conclude that, if (Lip) is satisfied and $\mathscr{E}_0$ small enough, then

$$\mathscr{E}(t) + \int_0^t \mathscr{D}\, d\tau \leq C\Big(1 + \|\langle v\rangle^2 f_0\|_{L^1_v(\mathbb{R}^3; L^\infty_x(\mathbb{R}^3))}\Big)\Big(\|u_0\|_{H^1}^2 + \iint f_0 |v|^2\, dv\, dx\Big) e^{CE_{0,0}}.$$

Last difficulty : how to ensure (Lip) ?

Lack of time integrability : we need to prove more decay for u.

Proving decay by Nash' argument

1. We need an energy balance : $\frac{d}{dt}\mathscr{L} + \mathscr{H} \le 0$;

2. the control of a 'lower order' functional $\mathscr{N} \le \mathscr{N}_0$;

3. an interpolation inequality: $\mathscr{L} \le C\mathscr{H}^{\theta}\mathscr{N}^{1-\theta}$ for some $\theta \in (0,1)$.

Setting $c_0 := C^{-1/\theta}\mathscr{N}_0^{1-1/\theta}$, we get

$$\frac{d}{dt}\mathscr{L} + c_0\mathscr{L}^{1/\theta} \le 0,$$

whence

$$\mathscr{L}(t) \le \mathscr{L}(0)\left(1 + \frac{1-\theta}{\theta}c_0\mathscr{L}^{\frac{1-\theta}{\theta}}t\right)^{-\frac{\theta}{1-\theta}}.$$

Application to the Navier-Stokes equations

The energy balance involves $\mathscr{L} = \|u\|_{L^2}^2$ and $\mathscr{H} = \|\nabla u\|_{L^2}^2$.

Interpolation inequality: $\|u\|_{L^2} \le C\|u\|_{L^1}^{2/5}\|\nabla u\|_{L^2}^{3/5}$.

If it is true that $\|u(t)\|_{L^1} \le C\|u_0\|_{L^1}$ then one can take $\mathscr{N} = \|u\|_{L^1}$ and get

$$\|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2\Big(1 + a_0 t\Big)^{-3/2} \quad \text{with} \quad a_0 = c_0\|u_0\|_{L^1}^{4/3}. \qquad (1)$$

Furthermore, from the energy balance,

$$\frac{d}{dt}\big(1 + a_0 t\big)^{\beta}\|u\|_{L^2}^2 + \big(1 + a_0 t\big)^{\beta}\|\nabla u\|_{L^2}^2 \le \beta a_0\big(1 + a_0 t\big)^{\beta-1}\|u\|_{L^2}^2.$$

Integrating and using (1) gives $\int_{\mathbb{R}_+}\big(1 + a_0 t\big)^{\beta}\|\nabla u\|_{L^2}^2\,dt \le C_\beta\|u_0\|_{L^2}^2$ for all $\beta < 3/2$.

One cannot take L^1 norm, but *for finite energy solutions*, we have:

$$\|u(t)\|_{\dot{B}^{-3/2}_{2,\infty}} \le C\Big(\|u_0\|_{\dot{B}^{-3/2}_{2,\infty}} + \|u_0\|_{L^2}^2\Big).$$

Application to (VNS)

Take $\mathscr{L} = E_0, E_1,\ \mathscr{H} = D_0, D_1,\ \mathscr{N} = \|u\|_{\dot{B}^{-3/2}_{2,\infty}}$ to get

$$(1+a_0 t)^{3/2+j} E_j(t) + \int_0^t (1+a_0\tau)^{\alpha+j} D_j \, d\tau \le C E_{j,0}, \qquad j = 0,1, \qquad \alpha < 3/2.$$

Since $u_t - \Delta u + \nabla P = -u \cdot \nabla u + \int f(v-u)\, dv$, we have

$$\|u\|_{L^\infty(\mathbb{R}_+; \dot{B}^{-\frac{3}{2}}_{2,\infty})} \le \|u_0\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}} + \left\| u \cdot \nabla u + \int f(v-u)\, dv \right\|_{L^1(\mathbb{R}_+; \dot{B}^{-\frac{3}{2}}_{2,\infty})}.$$

By embedding and Cauchy-Schwarz inequality,

$$\left\| \int f(v-u)\, dv \right\|_{L^2(\mathbb{R}_+; \dot{B}^{-3/2}_{2,\infty})} \lesssim \left\| \int f(v-u)\, dv \right\|_{L^2(\mathbb{R}_+; L^1)} \le \left(\|f\|_{L^\infty(\mathbb{R}_+; L^1_{x,v})} \iiint f|v-u|^2 \, dv\, dx \right)^{1/2} \le \sqrt{M_0 E_{0,0}}.$$

Difficulty: we only have L^2-in-time integrability, while we need L^1.

$\rightsquigarrow$ first prove a non optimal decay estimate, then bootstrap to get the desired exponent.

The Lipschitz bound

Use inequality $\|\nabla u\|_{L^\infty} \lesssim \|\nabla^2 u\|_{L^{3,1}}$ and elliptic regularity in $L^{3,1}$ for:

$$-\Delta u + \nabla P = -u_t - u\cdot\nabla u + \int f(v-u)\,dv, \qquad \operatorname{div} u = 0,$$

to get

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \lesssim \int_0^\infty \Big(\|u_t\|_{L^{3,1}} + \|u\cdot\nabla u\|_{L^{3,1}} + \Big\|\int f(v-u)\,dv\Big\|_{L^{3,1}}\Big)dt. \tag{2}$$

Use an interpolation inequality and the definition of D_1 and D_2:

$$\|u_t\|_{L^{3,1}} \lesssim \|u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{1/2} \lesssim t^{-1/4}(1+a_0 t)^{-(3/8)^-}\big((1+a_0 t)^{(3/2)^-}D_1\big)^{1/4}(tD_2)^{1/4}.$$

By Hölder inequality, we thus get,

$$\begin{aligned}\int_0^\infty \|u_t\|_{L^{3,1}}\,dt &\lesssim \Big(\int_0^\infty t^{-1/2}(1+a_0 t)^{-(3/4)^-}\,dt\Big)^{1/2}\Big(\int_0^\infty (1+a_0 t)^{(3/2)^-}D_1\,dt\Big)^{1/4}\Big(\int_0^\infty tD_2\,dt\Big)^{1/4}\\ &\le C_\alpha \mathscr{E}_0^{1/2}.\end{aligned}$$

The other terms of (2) may be treated similarly.

Extension of the main result to the critical regularity setting

• The main result holds (with the same decay estimates) if u_0 belongs to the critical Besov space $B^{1/2}_{2,1}$.

• Fujita-Kato type theorem: u_0 just belongs to $H^{1/2}$. A stronger localization of f_0 is needed (as in prior papers by Han Kwan et al).

In both case, the proof stems from the facts that:

• the regularity of u_0 and enough smallness suffice to have existence of a strong solution on a time interval of size 1;

• the velocity is instantaneously smoothed out, and one can find some $t_0 \in [0,1]$ for which $E_1(t_0)$ is small;

• we have uniqueness of the solution (more difficult in the $H^{1/2}$ case).