Barenblatt profiles and rarefaction waves in Euler Alignment system

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Nonlocal porous medium equation

The porous medium equation with the nonlocal pressure

$$\rho_t + (\rho u)_x = 0 \quad \text{with} \quad u_x = \Lambda^\alpha \rho$$

Biler, Karch, Monneu (2010)
Caffarelli, Vázquez (2011)
Biler, Imber, Karch (2015)
Stan, del Teso, Vázquez (2018)

Explicit solution - Barenblatt profile

The explicit self-similar solution of the porous medium equation with the nonlocal pressure

$$\rho_t + (\rho u)_x = 0 \quad \text{with} \quad u_x = \Lambda^\alpha \rho$$

if given by the formula

$$\rho(x,t) = \frac{1}{t^{1/(1+\alpha)}} \Phi\left(\frac{x}{t^{1/(1+\alpha)}}\right)$$

where the profile is given by the Getoor function

$$\rho(x,1) = \Phi(x) = K(\alpha,1)\left(1 - |x|^2\right)_+^{\frac{\alpha}{2}}$$

$$\Phi(x) = K(\alpha,1)\left(1-|x|^2\right)_+^{\frac{\alpha}{2}}$$

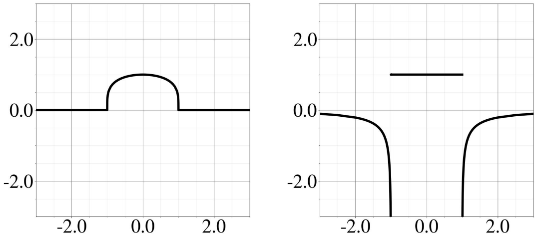
It was proved by Getoor (1961) that

$$\Lambda^\alpha \Phi(x) = \begin{cases} 1, & \text{for} \quad |x|<1, \\ H(x), & \text{for} \quad |x|>1, \end{cases}$$

where

$$H(x) = \left[\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(\frac{3+\alpha}{2}\right)\right]^{-1}\Gamma\left(\frac{1}{2}\right)\cdot |x|^{-1-\alpha}\cdot F\left(\frac{2+\alpha}{2},\frac{1+\alpha}{2},\frac{1+\alpha}{2},|x|^{-2}\right)$$

and $F(a,b,c,x)$ is a hypergeometric function.



Subject of this talk

Global-in-time solutions and their large time behavior of the following Euler Alignment system

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &u_t + u u_x = \int_{\mathbb{R}} \varphi(y-x)\big(u(y,t)-u(x,t)\big)\rho(y,t)\,dy, \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0,$$

with the singular interaction kernel

$$\varphi(x) = \frac{1}{|x|^{1+\alpha}} \qquad \text{for some} \quad \alpha \in (0,1)$$

and supplemented with **suitable initial conditions**.

The model

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x &= \int_{\mathbb{R}} \varphi(y-x)\big(u(y,t)-u(x,t)\big)\rho(x,t)\rho(y,t)\,\mathrm{d}y, \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0,$$

with a sufficiently regular interaction kernel $\varphi(z)$ (radial and positive) arises as the macroscopic realization of the Cucker and Smale (2007) agent model dynamics

$$x_i' = v_i, \qquad m v_i' = \frac{1}{N}\sum_{j=1}^{N} \varphi(|x_i - x_j|)(v_j - v_i),$$

which describes the collective motion of N individuals in particular alignment and flocking.

Motivation

$$\rho_t + (\rho u)_x = 0,$$
$$u_t + uu_x = \int_{\mathbb{R}} \varphi(y-x)\big(u(y,t) - u(x,t)\big)\rho(y,t)\,dy,$$

References (regular kernel)

- Tadmor and Tan (2014)
- Carrillo, Choi, Tadmor, Tan (2016)
- ...

References (singular kernel)

- Shvydkoy, Tadmor (2017) $\alpha \in (1,2)$, global smooth solutions
- Do, Kiselev, Ryzhik, Tan (2018), $\alpha \in (0,1)$, global smooth solutions
- Danchin, Mucha, Peszek, Wróblewski (2019), $\alpha \in (1,2)$, N-dimensional case
- Lear, Shvydkoy (2020), $\alpha \in (0,2)$, global smooth solutions, N-dimensional case
- ...

Compactly supported $\rho(x,t) \geq 0$?

The Euler alignment system

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &u_t + u u_x = \int_{\mathbb{R}} \frac{1}{|y-x|^{1+\alpha}} \big(u(y,t) - u(x,t)\big) \rho(y,t)\, \mathrm{d}y, \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0,$$

can be written, by using the relation

$$\big(u(y) - u(x)\big)\rho(y) = u(x)\big(\rho(x) - \rho(y)\big) - \big(u(x)\rho(x) - u(y)\rho(y)\big)$$

and the definition of the fractional Laplacian

$$\Lambda^\alpha f(x) = \int_{\mathbb{R}} \frac{f(x) - f(y)}{|y-x|^{\alpha+1}}\, \mathrm{d}y,$$

in the form

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + u(u_x - \Lambda^\alpha \rho) &= -\Lambda^\alpha(\rho u), \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0.$$

Equivalent formulation

System

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + u(u_x - \Lambda^\alpha \rho) &= -\Lambda^\alpha(\rho u), \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0,$$

can be expressed also in the following **formally equivalent** form

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ G_t + (Gu)_x &= 0, \\ u_x - \Lambda^\alpha \rho &= G, \end{aligned} \qquad x \in \mathbb{R}, \quad t > 0.$$

For $G = 0$, the system

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ G_t + (Gu)_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0. \\ u_x - \Lambda^\alpha \rho &= G, \end{aligned}$$

reduces to the porous medium equation with the nonlocal pressure

$$\rho_t + (\rho u)_x = 0 \quad \text{with} \quad u_x = \Lambda^\alpha \rho$$

Barenblatt profile

The explicit solution of the equation

$$\rho_t + (\rho u)_x = 0 \quad \text{with} \quad u_x = \Lambda^\alpha \rho$$

if given by the formula

$$\rho(x,t) = \frac{1}{t^{1/(1+\alpha)}} \Phi\left(\frac{x}{t^{1/(1+\alpha)}}\right)$$

where the profile is given by the Getoor function

$$\rho(x,1) = \Phi(x) = K(\alpha,1)\left(1 - |x|^2\right)_+^{\frac{\alpha}{2}}$$

Thus

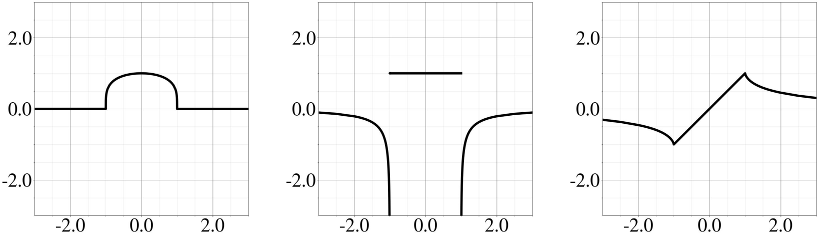
$$u(x,t) = U\left(\frac{x}{t^{1/(1+\alpha)}}\right),$$

where

$$u(x,1) = \partial_x^{-1}\Lambda^\alpha \Phi(x)$$

Self similiar profiles

$\Phi(x)$ $\qquad$ $\Lambda^{\alpha}\Phi(x)$ $\qquad$ $U(x) = \partial^{-1}\Lambda^{\alpha}\Phi(x)$



$$\rho(x,t) = \frac{1}{t^{1/(1+\alpha)}}\Phi\left(\frac{x}{t^{1/(1+\alpha)}}\right), \qquad u(x,t) = U\left(\frac{x}{t^{1/(1+\alpha)}}\right),$$

Problem description

We study two systems

$(\mathbf{U}\rho)$:

$$\rho_t + (\rho u)_x = 0,$$
$$u_t + u(u_x - \Lambda^\alpha \rho) = -\Lambda^\alpha(\rho u),$$

$(\mathbf{U}\rho\mathbf{G})$:

$$\rho_t + (\rho u)_x = 0,$$
$$G_t + (Gu)_x = 0,$$
$$u_x - \Lambda^\alpha \rho = G,$$

for $x \in \mathbb{R}$ and $t > 0$ with initial conditions

$$\rho_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \qquad u_0 \in L^\infty(\mathbb{R}), \qquad G_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}).$$

Weak solution of $(U\rho)$

Definition

A couple (u, ρ) is a weak solution of system $(U\rho)$ with initial conditions $\rho_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$ if

- $u \in L^\infty([0,\infty) \times \mathbb{R})$ and $\rho \in L^p\big([0,T], L^q(\mathbb{R})\big)$ for all $p, q \in (1,\infty)$ and $T > 0$,
- $u \in C\big([0,T], L^q([-R,R])\big)$ for each $T > 0$, $R > 0$, and $q \in \big(1/(1-\alpha), \infty\big)$,
- The following equations holds true

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \rho\varphi_t \, dx\, dt - \int_{\mathbb{R}} \rho_0 \varphi(\cdot,0)\, dx + \int_0^\infty \int_{\mathbb{R}} \rho u \varphi_x \, dx\, dt = 0,\\
&\int_0^\infty \int_{\mathbb{R}} u\varphi_t \, dx\, dt - \int_{\mathbb{R}} u_0 \varphi(\cdot,0)\, dx + \int_0^\infty \int_{\mathbb{R}} u\rho\Lambda^\alpha \varphi \, dx\, dt\\
&\qquad + \int_0^\infty \int_{\mathbb{R}} u(u_x - \Lambda^\alpha \rho)\varphi \, dx\, dt = 0
\end{aligned}$$

for each $\varphi \in C_c^\infty\big(\mathbb{R} \times [0,\infty)\big)$.

Definition

A triple (u, ρ, G) is a weak solution of system (UρG) with initial conditions $\rho_0, G_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ if

- $u \in L^\infty([0,\infty) \times \mathbb{R})$ and $\rho, G \in L^p\big([0,T], L^q(\mathbb{R})\big)$ for all $p, q \in (1,\infty)$ and $T > 0$,
- $u \in C\big([0,T], L^q([-R,R])\big)$ for each $T > 0$, $R > 0$ and $q \in \big(1/(1-\alpha), \infty\big)$,
- The following equations hold true

$$\int_0^\infty \int_{\mathbb{R}} \rho\varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} \rho_0\varphi(\cdot,0)\,\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \rho u\varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0,$$
$$\int_0^\infty \int_{\mathbb{R}} G\varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} G_0\varphi(\cdot,0)\,\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} Gu\varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0,$$
$$-\int_0^\infty \int_{\mathbb{R}} u\varphi_x \,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \rho\Lambda^\alpha\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}} G\varphi \,\mathrm{d}x\,\mathrm{d}t$$

for each $\varphi \in C_c^\infty\big(\mathbb{R} \times [0,\infty)\big)$.

Existence of weak solutions

Existence of solutions for $\alpha \in (0,1)$

A solution is obtained as a limit of solutions to the regularized system

$$\begin{aligned} \rho_t^\varepsilon + (\rho^\varepsilon u^\varepsilon)_x &= \varepsilon \rho_{xx}^\varepsilon, \\ G_t^\varepsilon + (G^\varepsilon u^\varepsilon)_x &= \varepsilon G_{xx}^\varepsilon, \qquad\qquad x \in \mathbb{R}, \quad t > 0, \\ u_x^\varepsilon &= G^\varepsilon + \Lambda^\alpha \rho^\varepsilon, \end{aligned}$$

with $\varepsilon > 0$ and supplemented with the initial conditions

$$\begin{aligned} \rho^\varepsilon(x,0) &= \rho_0(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \\ G^\varepsilon(x,0) &= G_0(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \end{aligned}$$

satisfying

$$0 \leqslant G_0(x) \leqslant a\rho_0(x) \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and some fixed } a > 0.$$

Existence of solutions

Theorem

Assume that there exist $a > 0$ *such that*

$$0 \leqslant G_0(x) \leqslant a\rho_0(x) \quad \text{for almost all} \quad x \in \mathbb{R}.$$

There exist (ρ, u, G) *which is a weak solution with the following properties for each* $t > 0$*:*

1. *The estimate* $\|u(\cdot,t)\|_\infty \leqslant \|u_0\|_\infty$.
2. *Comparison principle:* $\rho(x,t) \geqslant 0, \quad 0 \leqslant G(x,t) \leqslant a\rho(x,t)$
3. L^p*-estimates and decay*

$$\frac{1}{a}\|G(\cdot,t)\|_p \leqslant \|\rho(\cdot,t)\|_p \leqslant \|\rho_0\|_p \quad \text{for each} \quad p \in [1,\infty].$$

and

$$\frac{1}{a}\|G(\cdot,t)\|_p \leqslant \|\rho(\cdot,t)\|_p \leqslant CM_\rho^{\frac{2+p\alpha}{2p+2\alpha}} t^{\left(-1+\frac{1}{p}\right)\frac{1}{1+\alpha}} \quad \text{for each} \quad p \in (2,\infty).$$

We multiply first equation by $p\,(\rho^\varepsilon)^{p-1}$ and integrate

$$\frac{d}{dt}\int_{\mathbb{R}} (\rho^\varepsilon)^p \,\mathrm{d}x + p\int_{\mathbb{R}} (\rho^\varepsilon u^\varepsilon)_x\,(\rho^\varepsilon)^{p-1}\,\mathrm{d}x = \varepsilon p\int_{\mathbb{R}} \rho^\varepsilon_{xx}\,(\rho^\varepsilon)^{p-1}\,\mathrm{d}x.$$

Next, we integrate the middle term twice by parts and we substitute the relation $u^\varepsilon_x = G^\varepsilon + \Lambda^\alpha \rho^\varepsilon$ to obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} (\rho^\varepsilon)^p \,\mathrm{d}x =& -(p-1)\int_{\mathbb{R}} (\rho^\varepsilon)^p\,\Lambda^\alpha \rho^\varepsilon \,\mathrm{d}x\\ & -(p-1)\int_{\mathbb{R}} (\rho^\varepsilon)^p\,G^\varepsilon\,\mathrm{d}x - \varepsilon p(p-1)\int_{\mathbb{R}} (\rho^\varepsilon_x)^2\,(\rho^\varepsilon)^{p-2}\,\mathrm{d}x.\end{aligned}$$

By the Strook-Varopoulos inequality:

$$\frac{d}{dt}\int_{\mathbb{R}} (\rho^\varepsilon)^p \,\mathrm{d}x \leqslant -C\,\frac{p^2-1}{(p+1)^2}\int_{\mathbb{R}} \left(\Lambda^{\frac{\alpha}{2}}(\rho^\varepsilon)^{\frac{p+2}{2}}\right)^2 \mathrm{d}x$$

Next, for $p>2$, we use

$$\int_{\mathbb{R}} \left(\Lambda^{\frac{\alpha}{2}}\rho^\varepsilon)^{\frac{p+2}{2}}\right)^2 \geqslant C\,\frac{1}{M_\rho^{\frac{2+p\alpha}{p-1}}}\left(\int_{\mathbb{R}} (\rho^\varepsilon)^p\,\mathrm{d}x\right)^{\frac{p+1+\alpha}{p-1}}.$$

Estimates of u^ε for $\alpha \in (0,1)$

Denoting

$$\partial_x^{-1} v(x) = \int_{-\infty}^{x} v(y)\,\mathrm{d}y,$$

the equation

$$u_x^\varepsilon - \Lambda^\alpha \rho^\varepsilon = G^\varepsilon$$

can written in the form

$$u^\varepsilon = \partial_x^{-1} G^\varepsilon + \partial_x^{-1} \Lambda^\alpha \rho^\varepsilon,$$

under the assumption that $\lim\limits_{x\to-\infty} u^\varepsilon(x) = 0$.

On the other hand, we also have

$$\Lambda^{-\alpha}\partial_x u^\varepsilon = \Lambda^{-\alpha} G^\varepsilon + \rho^\varepsilon$$

Lemma

For arbitrary $p \in (1, 1/\alpha)$ and $q \in (p, \infty)$ such that $1/q = 1/p - \alpha$, if $G \in L^p(\mathbb{R})$, $\rho \in L^q(\mathbb{R})$, and $u_x = G + \Lambda^\alpha \rho$ in the sense of tempered distributions then $\Lambda^{1-\alpha} u \in L^q(\mathbb{R})$ together with the estimate

$$\|\Lambda^{1-\alpha} u\|_q \leqslant C\big(\|G\|_p + \|\rho\|_q\big)$$

The families $\{\rho^\varepsilon\}_{\varepsilon>0}, \{G^\varepsilon\}_{\varepsilon>0}$ are bounded in the spaces $C\big([0,\infty), L^p(\mathbb{R})\big)$
For all $r,p \in (1,\infty)$ and each $T>0$

$$\begin{aligned} \rho^{\varepsilon_k} &\to \rho, \\ G^{\varepsilon_k} &\to G, \end{aligned} \quad \text{weakly in } L^r\big([0,T], L^p(\mathbb{R})\big).$$

We pass to the limit with $\varepsilon \to 0$

$$-\int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon \varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} \rho_0 \varphi(\cdot,0)\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon u^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t = \varepsilon \int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon \varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t,$$

$$-\int_0^\infty \int_{\mathbb{R}} G^\varepsilon \varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} G_0 \varphi(\cdot,0)\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}} G^\varepsilon u^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t = \varepsilon \int_0^\infty \int_{\mathbb{R}} G^\varepsilon \varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t,$$

$$-\int_0^\infty \int_{\mathbb{R}} u^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon \Lambda^\alpha \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}} G^\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t$$

for each $\varphi \in C_c^\infty\big(\mathbb{R}\times[0,\infty)\big)$

For each $q \in (1,\infty)$ and all $R > 0$ the sequence u^ε is uniformly bounded in

$$\{u^\varepsilon\}_{\varepsilon>0} \subseteq L^\infty\big([0,\infty),\ H_q^{1-\alpha}([-R,R])\big)$$

and by Aubin-Lions lemma

$$u^{\varepsilon_k} \to u \quad \text{strongly in} \quad C\big([0,T], L^q([-R,R])\big)$$

We pass to the limit with $\varepsilon \to 0$

$$-\int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon \varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} \rho_0 \varphi(\cdot,0)\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon u^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t = \varepsilon \int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon \varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t,$$

$$-\int_0^\infty \int_{\mathbb{R}} G^\varepsilon \varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} G_0 \varphi(\cdot,0)\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}} G^\varepsilon u^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t = \varepsilon \int_0^\infty \int_{\mathbb{R}} G^\varepsilon \varphi_{xx}\,\mathrm{d}x\,\mathrm{d}t,$$

$$-\int_0^\infty \int_{\mathbb{R}} u^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \rho^\varepsilon \Lambda^\alpha \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}} G^\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t$$

for each $\varphi \in C_c^\infty\big(\mathbb{R} \times [0,\infty)\big)$

Large time asymptotics

Scaling

For $\lambda > 0$, denote

$$\rho^{\lambda}(x,t) \equiv \lambda \rho(\lambda x, \lambda t),$$

$$G^{\lambda}(x,t) \equiv \lambda G(\lambda x, \lambda t),$$

$$u^{\lambda}(x,t) \equiv u(\lambda x, \lambda t).$$

Scaling

The rescaled solutions

$$\rho^\lambda(x,t) \equiv \lambda\rho(\lambda x, \lambda t), \qquad G^\lambda(x,t) \equiv \lambda G(\lambda x, \lambda t), \qquad u^\lambda(x,t) \equiv u(\lambda x, \lambda t).$$

satisfy the following systems (in the new variables $y = x/\lambda$ and $s = t/\lambda$)

$$\begin{aligned} \rho_s^\lambda + (u^\lambda \rho^\lambda)_y &= 0, \\ u_s^\lambda + u^\lambda(u_y^\lambda - \lambda^{-\alpha}\Lambda^\alpha \rho^\lambda) &= -\lambda^{-\alpha}\Lambda^\alpha(u^\lambda \rho^\lambda). \end{aligned}$$

as well as

$$\begin{aligned} \rho_s^\lambda + (u^\lambda \rho^\lambda)_y &= 0, \\ G_s^\lambda + (u^\lambda G^\lambda)_y &= 0, \\ u_y^\lambda &= G^\lambda + \lambda^{-\alpha}\Lambda^\alpha \rho^\lambda. \end{aligned}$$

We consider system

$$\begin{aligned} \rho_s^\lambda + (u^\lambda \rho^\lambda)_y &= 0, \\ G_s^\lambda + (u^\lambda G^\lambda)_y &= 0, \qquad\qquad x \in \mathbb{R}, \quad t > 0, \\ u_y^\lambda &= G^\lambda + \lambda^{-\alpha} \Lambda^\alpha \rho^\lambda. \end{aligned}$$

supplemented with the initial conditions

$$\rho^\lambda(x,0) = \lambda \rho_0(\lambda x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$$
$$G^\lambda(x,0) = \lambda G_0(\lambda x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$$

satisfying

$$0 \leqslant b\rho_0(x) \leqslant G_0(x) \leqslant a\rho_0(x) \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and some fixed } a, b > 0.$$

Theorem

Assume the initial condition satisfies

$$0 \leqslant b\rho_0(x) \leqslant G_0(x) \leqslant a\rho_0(x) \quad \text{for all} \quad x \in \mathbb{R}$$

and some $a > 0$ *and* $b > 0$*. The following properties hold true for all* $t > 0$*:*

1. *The inequality is satisfied*

$$b\rho(x,t) \leqslant G(x,t) \leqslant a\rho(x,t) \quad \text{for all} \quad x \in \mathbb{R}, \ t > 0.$$

2. *There exist a constant* $C > 0$ *independent of* p *and* t *such that for all* $p \in [1, \infty]$ *and* $t > 0$

$$\|\rho(\cdot,t)\|_p \leqslant CM_\rho^{\frac{1}{p}} t^{-1+\frac{1}{p}} \quad \text{and} \quad \|G(\cdot,t)\|_p \leqslant CM_G^{\frac{1}{p}} t^{-1+\frac{1}{p}}.$$

The considered system

$$\begin{aligned} \rho_t^\varepsilon + (\rho^\varepsilon u^\varepsilon)_x &= \varepsilon \rho_{xx}^\varepsilon, \\ G_t^\varepsilon + (G^\varepsilon u^\varepsilon)_x &= \varepsilon G_{xx}^\varepsilon, \\ u_x^\varepsilon &= G^\varepsilon + \Lambda^\alpha \rho^\varepsilon, \end{aligned}$$

The estimate

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (\rho^\varepsilon)^p \, \mathrm{d}x = &- (p-1) \int_{\mathbb{R}} (\rho^\varepsilon)^p \, \Lambda^\alpha \rho^\varepsilon \, \mathrm{d}x \\ &- (p-1) \int_{\mathbb{R}} (\rho^\varepsilon)^p \, G^\varepsilon \, \mathrm{d}x - \varepsilon p(p-1) \int_{\mathbb{R}} (\rho_x^\varepsilon)^2 (\rho^\varepsilon)^{p-2} \, \mathrm{d}x. \end{aligned}$$

Now, by the Hölder inequality with the indices $q_1 = p$ and $q_2 = \frac{p}{p-1}$, we obtain

$$\int_{\mathbb{R}} (\rho^\varepsilon)^p \, \mathrm{d}x = \int_{\mathbb{R}} (\rho^\varepsilon)^{\frac{1}{p}} (\rho^\varepsilon)^{p-\frac{1}{p}} \, \mathrm{d}x \leqslant M_\rho^{\frac{1}{p}} \left(\int_{\mathbb{R}} (\rho^\varepsilon)^{p+1} \, \mathrm{d}x \right)^{\frac{p-1}{p}}$$

Scaling

Lemma

There exist numbers $C_1 = C_1(p, \rho_0) > 0$ *and* $C_2 = C_2(p, G_0) > 0$ *such that for all* $\lambda > 0$ *and all* $t > 0$ *the inequalities hold true*

$$\|\rho^\lambda(\cdot, t)\|_p \leqslant C_1 t^{-1+1/p} \quad \text{and} \quad \|G^\lambda(\cdot, t)\|_p \leqslant C_2 t^{-1+1/p} \quad \text{and} \quad \|u^\lambda(\cdot, t)\|_\infty \leqslant \|u_0\|_\infty.$$

Proof.

By the decay estimates, we obtain

$$\|\rho^\lambda(\cdot, t)\|_p \leqslant \|\lambda\rho(\lambda\cdot, \lambda t)\|_p \leqslant (\lambda t)^{-1+1/p}\lambda^{-1+1/p}\|\rho_0(\lambda\cdot)\|_p \leqslant t^{-1+1/p}\|\rho_0\|_p.$$

□

For each $t_1, t_2 > 0$ and all $r, p \in (1, \infty)$ the families $\{\rho^\lambda\}_\lambda, \{G^\lambda\}_\lambda$ are bounded in the space $L^r\big([t_1, t_2],\, L^p(\mathbb{R})\big)$ and

$$\begin{matrix} \rho^{\lambda_k} \to \rho \\ G^{\lambda_k} \to G \end{matrix} \quad \text{weakly in } \; L^r\big([t_1, t_2], L^p(\mathbb{R})\big),$$

For each $q \in (1, \infty)$ each $t_1, t_2 > 0$ and all $R > 0$ the sequence u^λ is uniformly bounded in

$$\{u^\lambda\}_\lambda \subseteq L^\infty\big([t_1, t_2),\, H_q^{1-\alpha}([-R, R])\big)$$

and by Aubin-Lions lemma

$$u^\lambda \to u \quad \text{strongly in} \quad C\big([t_1, t_2], L^q([-R, R])\big)$$

$$
\begin{aligned}
-\int_0^\infty \int_{\mathbb{R}} \rho^\lambda \varphi_s \, dy \, ds - \int_{\mathbb{R}} \rho_0^\lambda \varphi(\cdot,0) \, dy - \int_0^\infty \int_{\mathbb{R}} \rho^\lambda u^\lambda \varphi_y \, dy \, ds = 0,\\
-\int_0^\infty \int_{\mathbb{R}} u^\lambda \varphi_s \, dy \, ds - \int_{\mathbb{R}} u_0^\lambda \varphi(\cdot,0) \, dy + \lambda^{-\alpha} \int_0^\infty \int_{\mathbb{R}} u^\lambda \rho^\lambda \Lambda^\alpha \varphi \, dy \, ds \\
+ \int_0^\infty \int_{\mathbb{R}} u^\lambda (u_y^\lambda - \lambda^{-\alpha} \Lambda^\alpha \rho^\lambda) \varphi \, dy \, ds = 0
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_0^\infty \int_{\mathbb{R}} \rho^\lambda \varphi_s \, dy \, ds - \int_{\mathbb{R}} \rho_0^\lambda \varphi(\cdot,0) \, dy - \int_0^\infty \int_{\mathbb{R}} \rho^\lambda u^\lambda \varphi_y \, dy \, ds = 0,\\
-\int_0^\infty \int_{\mathbb{R}} G^\lambda \varphi_s \, dy \, ds - \int_{\mathbb{R}} G_0^\lambda \varphi(\cdot,0) \, dy - \int_0^\infty \int_{\mathbb{R}} G^\lambda u^\lambda \varphi_y \, dy \, ds = 0,\\
-\int_0^\infty \int_{\mathbb{R}} u^\lambda \varphi_y \, dy \, ds - \lambda^{-\alpha} \int_0^\infty \int_{\mathbb{R}} \rho^\lambda \Lambda^\alpha \varphi \, dy \, ds - \int_0^\infty \int_{\mathbb{R}} G^\lambda \varphi \, dy \, ds = 0,
\end{aligned}
$$

for each $\varphi \in C_c^\infty\big(\mathbb{R} \times [0,\infty)\big)$.

Theorem

Let (u, ρ, G) be a weak solution corresponding to the initial condition satisfying

$$0 \leqslant b\rho_0(x) \leqslant G_0(x) \leqslant a\rho_0(x) \quad \textit{for all} \quad x \in \mathbb{R}$$

and some $a > 0$ and $b > 0$. For each $r, p \in (1, \infty)$ and $q \in \big(1/\alpha, \infty\big)$

$$\begin{array}{c} \rho^{\lambda_k} \to \overline{\rho} \\ G^{\lambda_k} \to \overline{G} \end{array} \quad \textit{weakly in } L^r\big([t_1, t_2], L^p(\mathbb{R})\big),$$

$$u^{\lambda_k} \to \overline{u} \quad \textit{strongly in } C\big([t_1, t_2], L^q([-R, R])\big),$$

for all $0 < t_1 < t_2$ and $R > 0$, where $\big(\overline{u}, \overline{\rho}, \overline{G}\big)$ is a weak solution to systems

$$\begin{aligned} \overline{\rho}_t + (\overline{u}\overline{\rho})_x &= 0, \\ \overline{G}_t + (\overline{u}\,\overline{G})_x &= 0, \\ \overline{u}_x &= \overline{G}, \end{aligned} \qquad \begin{aligned} \overline{\rho}_t + (\overline{u}\overline{\rho})_x &= 0, \\ \overline{u}_t + \overline{u}\overline{u}_x &= 0, \end{aligned}$$

with the the initial conditions

$$\overline{\rho}(x, 0) = M_\rho \delta_0, \qquad \overline{G}(x, 0) = M_G \delta_0,$$

Limit profiles

The limit profiles are given by the explicit formulas

$$\overline{\rho}(x,t) = \begin{cases} 0, & x \leqslant 0, \\ \frac{1}{t}, & 0 < x \leqslant M_\rho t, \\ 0, & x > M_\rho t, \end{cases} \qquad \overline{u}(x,t) = \begin{cases} 0, & x \leqslant 0, \\ \frac{x}{t}, & 0 < x \leqslant M_G t, \\ M_G, & x > M_G t. \end{cases}$$

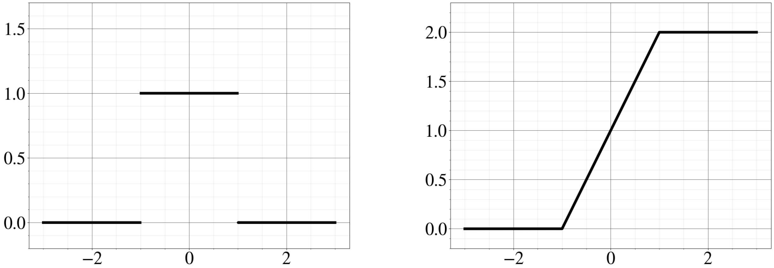
$$\overline{G}(x,t) = \begin{cases} 0, & x \leqslant 0, \\ \frac{1}{t}, & 0 < x \leqslant M_G t, \\ 0, & x > M_G t, \end{cases}$$

Limit profiles

The rescalled solution

$$\lambda\rho(\lambda x, \lambda t) \quad \text{and} \quad u(\lambda x, \lambda t)$$

converges for large λ to the explicit functions $\overline{u}(x,t)$ and $\overline{\rho}(x,t)$ illustrated on the figures.



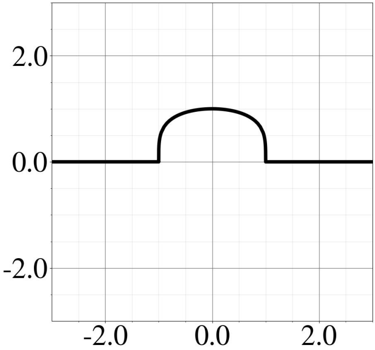
Initial conditions satisfy

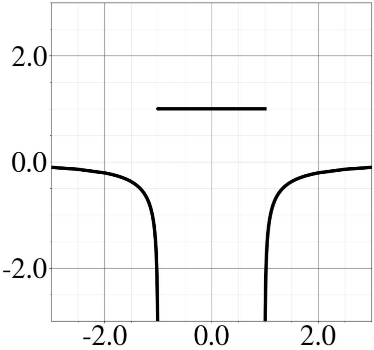
$$b\rho_0(x) \leqslant G_0(x) \leqslant a\rho_0(x)$$

Since $G_0 = (u_0)_x - \Lambda^\alpha \rho_0$

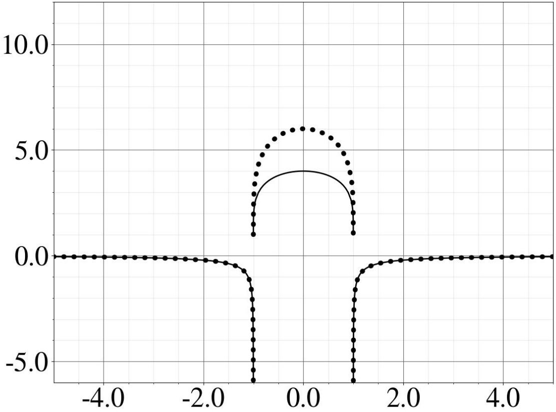
$$b\rho_0(x) + \Lambda^\alpha \rho_0(x) \leqslant (u_0)_x(x) \leqslant a\rho_0(x) + \Lambda^\alpha \rho_0(x)$$

Plots of the Getoor function $\Phi(x)$ and of $\Lambda^\alpha \Phi(x)$

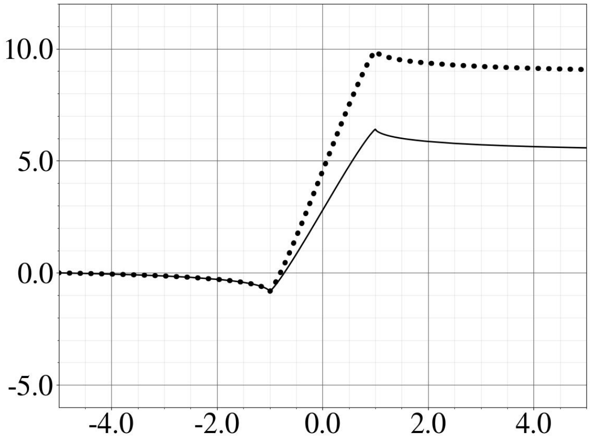




Plots of $a\Phi(x) + \Lambda^\alpha \Phi(x)$ for $a = 5$ and $a = 1$



Profile of $U(x)$



Thank you for your attention