Fokker–Planck equations and functional inequalities for heavy tailed probability densities

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The classical Fokker–Planck equation

We consider the following Cauchy problem for the classical Fokker–Planck equation

$$\begin{cases} \partial_t f(v,t) = \partial_v^2 f(v,t) + \partial_v \left(v f(v,t) \right) & v \in \mathbb{R},\ t > 0 \\ f(v,0) = f_0(v) \in L^1(\mathbb{R}) \end{cases}$$

where f_0 is a probability density.

This equation has a unique stationary solution of unit mass, given by the Gaussian distribution (which is called in this framework the **Maxwellian distribution**)

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}, \quad v \in \mathbb{R}$$

For $f_0 \in L^1(\mathbb{R})$ the solution of the Cauchy problem is classical and has the form

$$f(v,t) = f_{0,\alpha(t)} * M_{\beta(t)}(v)$$

where

$$g_a(v) = \frac{1}{a} g\left(\frac{v}{a}\right)$$

and

$$\alpha(t) = e^{-t}, \quad \beta(t) = \sqrt{1 - e^{-2t}}$$

Remark: if f_0 is a probability density, the solution $f(t)$ is a probability density for all times.

The functional inequalities

A probability density f_∞ on $\mathcal{I} \subseteq \mathbb{R}$ is said to satisfy a **Poincaré inequality** if for any smooth function φ on $\mathcal{I}$

$$\int_{\mathcal{I}} \left(\varphi(v) - \left(\int_{\mathcal{I}} \varphi(v)\, f_\infty(v)\, dv \right) \right)^2 f_\infty(v)\, dv \le C \int_{\mathcal{I}} (\varphi'(v))^2\, f_\infty(v)\, dv$$

Likewise, f_∞ is said to satisfy a **logarithmic Sobolev inequality** if, for any smooth function φ on $\mathcal{I}$

$$\int_{\mathcal{I}} \varphi^2(v) \log \varphi^2(v)\, f_\infty(v)\, dv - \left(\int_{\mathcal{I}} \varphi^2(v)\, f_\infty(v)\, dv \right) \log \left(\int_{\mathcal{I}} \varphi^2(v)\, f_\infty(v)\, dv \right)$$
$$\le C \int_{\mathcal{I}} (\varphi'(v))^2\, f_\infty(v)\, dv.$$

It is well known that the Gaussian function satisfies both Poincaré (Nash 59, Chernoff 81, ...) and Log–Sobolev inequalities (Gross 75)

$$\int_{\mathbb{R}} \left(\varphi(v) - \left(\int_{\mathbb{R}} \varphi(v)\, M(v)\, dv \right) \right)^2 M(v)\, dv \le \int_{\mathbb{R}} (\varphi'(v))^2\, M(v)\, dv$$

$$\int_{\mathbb{R}} \varphi^2(v) \log \varphi^2(v)\, M(v)\, dv - \left(\int_{\mathbb{R}} \varphi^2(v)\, M(v)\, dv \right) \log \left(\int_{\mathbb{R}} \varphi^2(v)\, M(v)\, dv \right) \le 2 \int_{\mathbb{R}} (\varphi'(v))^2\, M(v)\, dv$$

Which relation between Fokker–Planck and functional inequalities ?

1. Proof of Poincaré inequality for the Gaussian distribution, exploiting FP

$$\int_{\mathbb{R}} \left(\varphi(v) - \int_{\mathbb{R}} \varphi(v) M(v)\, dv \right)^2 M(v)\, dv \le \int_{\mathbb{R}} (\varphi(v) - \varphi(0))^2 M(v)\, dv$$

$$= \int_{\mathbb{R}} \left(\int_0^v \varphi'(s)\, ds \right)^2 M(v)\, dv = \int_{\mathbb{R}} \left(\int_0^1 \varphi'(vt) v\, dt \right)^2 M(v)\, dv$$

$$\overset{\text{Jensen}}{\le} \int_{\mathbb{R}} \left(\int_0^1 (\varphi'(vt)v)^2\, dt \right) M(v)\, dv = \int_{\mathbb{R}} vM(v) \left(\int_0^1 (\varphi'(vt))^2 v\, dt \right) dv$$

$$\overset{\partial_v M(v) + vM(v) = 0}{=} \int_{\mathbb{R}} -\partial_v M(v) \left(\int_0^1 (\varphi'(vt))^2 v\, dt \right) dv$$

$$= \left[-M(v) \left(\int_0^1 (\varphi'(vt))^2 v\, dt \right) \right]_{-\infty}^{+\infty} + \int_{\mathbb{R}} M(v) \partial_v \left(\int_0^1 (\varphi'(vt))^2 v\, dt \right) dv$$

$$\overset{\text{first term is non positive}}{\le} \int_{\mathbb{R}} M(v) \left(\int_0^1 \partial_v \left((\varphi'(vt))^2 v \right) dt \right) dv$$

$$\overset{\partial_v(vf(vt)) = \partial_t(tf(vt))}{=} \int_{\mathbb{R}} M(v) \left(\int_0^1 \partial_t \left((\varphi'(vt))^2 t \right) dt \right) dv$$

$$= \int_{\mathbb{R}} (\varphi'(v))^2 M(v)\, dv$$

Which relation between Fokker–Planck and functional inequalities ?

2. Log–Sobolev inequality and L^1 convergence to the steady state

For any initial density f_0 such that $\int_{\mathbb{R}} f_0(v)(1 + v^2 + \log f_0(v))dv < \infty$ it is well known that the corresponding solution of the classical Fokker–Planck equation (which is itself a probability density) converges in $L^1(\mathbb{R})$ at an exponential rate towards the stationary state $M(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$.

We recall here the ingredients of a proof (Toscani 99)

Let f and g two densities on $\mathbb{R}$.

- the **Shannon entropy** of f relative to g is

$$H(f,g) = \int_{\mathbb{R}} f(v) \log \frac{f(v)}{g(v)}\, dv$$

- the **Fisher information** of f relative to g is

$$I(f,g) = 4 \int_{\mathbb{R}} \left(\partial_v \sqrt{\frac{f(v)}{g(v)}} \right)^2 g(v)\, dv$$

- It is well known that $H(f(t), M)$ is decreasing along the solution $f(t)$ of the Fokker–Planck equation and more precisely (McKean 66)

$$\frac{d}{dt} H(f(t), M) = -I(f(t), M).$$

Let us rewrite the terms:

$$H(f(t), M) = \int_{\mathbb{R}} f(t) \log \frac{f(t)}{M}\, dv$$

$$= \int_{\mathbb{R}} \sqrt{\frac{f(t)}{M}}^2 \log \sqrt{\frac{f(t)}{M}}^2 M\, dv - \left(\int_{\mathbb{R}} \sqrt{\frac{f(t)}{M}}^2 M dv \right) \log \left(\int_{\mathbb{R}} \sqrt{\frac{f(t)}{M}}^2 M dv \right)$$

and

$$I(f(t), M) = 4 \int_{\mathbb{R}} \left(\partial_v \sqrt{\frac{f(t)}{M}} \right)^2 M\, dv$$

- By **Log-Sobolev inequality** for the Gaussian density M applied to $\varphi = \sqrt{\frac{f(t)}{M}}$

$$\int_{\mathbb{R}} \sqrt{\frac{f(t)}{M}}^2 \log \sqrt{\frac{f(t)}{M}}^2 M\,dv - \left(\int_{\mathbb{R}} \sqrt{\frac{f(t)}{M}}^2 Mdv \right) \log \left(\int_{\mathbb{R}} \sqrt{\frac{f(t)}{M}}^2 Mdv \right)$$

$$\leq 2 \int_{\mathbb{R}} \left(\partial_v \sqrt{\frac{f(t)}{M}} \right)^2 M\,dv = \frac{1}{2} \cdot 4 \int_{\mathbb{R}} \left(\partial_v \sqrt{\frac{f(t)}{M}} \right)^2 M\,dv$$

we get

$$H(f(t), M) \leq \frac{1}{2} I(f(t), M)$$

- we get exponential decay in relative entropy

$$\frac{d}{dt} H(f(t), M) = -I(f(t), M) \leq -2H(f(t), M)$$

$$\Rightarrow H(f(t), M) \leq e^{-2t} H(f_0, M).$$

- Last, Csiszár–Kullback inequality (Csiszár 63 – Kullback 59)

$$\|f-g\|_{L^1}^2 \le 2H(f,g)$$

permits to prove that $f(t)$ converges exponentially fast in L^1 to the Gaussian density

$$\|f(t)-M\|_{L^1} \le \sqrt{2}e^{-t}\sqrt{H(f_0,M)}.$$

- The assumption

$$\int_{\mathbb{R}} f_0(v)(1+v^2+\log f_0(v))dv < \infty$$

implies

$$\begin{aligned} H(f_0,M) &= \int_{\mathbb{R}} f_0(v)\log\frac{f_0(v)}{M(v)}\,dv \\ &= \int_{\mathbb{R}} f_0(v)\left(\log f_0(v) - \log\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}\right)\right)\,dv \\ &= C\int_{\mathbb{R}} f_0(v)(1+v^2+\log f_0(v))dv < \infty \end{aligned}$$

Also the Poincaré inequality allows us to prove exponential convergence in L^1 to the Gaussian stationary state M for the solution of the Cauchy problem with initial density f_0 (Markowich, Villani 2000) under a different, more restrictive assumption on f_0

Which relation between Fokker–Planck and functional inequalities ?

3. Poincaré inequality and convergence in $L^2(M(v)\,dv)$ for the adjoint problem

The adjoint equation

Letting $F(t,v) = \dfrac{f(t,v)}{M(v)}$, and recalling that the Gaussian density M **is a stationary state for the Fokker–Planck equation**, so

$$\partial_v M(v) + vM(v) = 0, \quad v \in \mathbb{R},$$

it can be easily proved that F satisfies the adjoint equation

$$\partial_t F(v,t) = \partial_v^2 F(v,t) - v\partial_v F(v,t), \quad v \in \mathbb{R},\ t > 0$$

and the corresponding Cauchy problem is

$$F(v,0) = \frac{f_0(v)}{M(v)}$$

We expect that $F(t)$ converges exponentially fast to 1.

We replace the Shannon entropy of f relative to the stationary state M by the square of a $L^2(Mdv)$ distance between F and 1

$$\int_{\mathbb{R}} f(t) \log \frac{f(t)}{M}\, dv \Rightarrow \int_{\mathbb{R}} (F(t) - 1)^2\, M\, dv$$

- We can compute

$$\frac{d}{dt} \int_{\mathbb{R}} (F(t) - 1)^2\, M\, dv = -2 \int_{\mathbb{R}} (\partial_v F(t))^2\, Mdv$$

Now:

$$\int_{\mathbb{R}} (F(t) - 1)^2\, M\, dv = \int_{\mathbb{R}} \left(\frac{f(t)}{M} - \left(\int_{\mathbb{R}} \frac{f(t)}{M}\, M\, dv \right) \right)^2 Mdv$$

$$\int_{\mathbb{R}} (\partial_v F(t))^2\, Mdv = \int_{\mathbb{R}} \left(\partial_v \frac{f(t)}{M} \right)^2 Mdv$$

So, by **Poincaré inequality** for the Gaussian M applied to $\varphi = \frac{f(t)}{M}$

$$\int_{\mathbb{R}} \left(\frac{f(t)}{M} - \left(\int_{\mathbb{R}} \frac{f(t)}{M}\, M\, dv \right) \right)^2 Mdv \leq \int_{\mathbb{R}} \left(\partial_v \frac{f(t)}{M} \right)^2 Mdv$$

we get

$$\int_{\mathbb{R}} (F(t) - 1)^2\, M\, dv \leq \int_{\mathbb{R}} (\partial_v F(t))^2\, Mdv$$

- so

$$\frac{d}{dt}\int_{\mathbb{R}} (F(t)-1)^2\, M\, dv \leq -2 \int_{\mathbb{R}} (F(t)-1)^2\, M\, dv$$

- This implies exponential convergence

$$\int_{\mathbb{R}} (F(t)-1)^2\, M\, dv \leq e^{-2t} \int_{\mathbb{R}} (F_0-1)^2\, M\, dv$$

- and on the original solution

$$\int_{\mathbb{R}} \frac{(f(t)-M)^2}{M}\, dv \leq e^{-2t} \int_{\mathbb{R}} \frac{(f_0-M)^2}{M}\, dv$$

- this convergence result still implies exponential L^1 convergence since

$$\int_{\mathbb{R}} |f(t)-M|\, dv \leq \left(\int_{\mathbb{R}} \frac{(f(t)-M)^2}{M}\, dv\right)^{1/2} \Bigg(\overbrace{\int_{\mathbb{R}} M\, dv}^{=1}\Bigg)^{1/2}$$
$$\leq e^{-t} \left(\int_{\mathbb{R}} \frac{(f_0-M)^2}{M}\, dv\right)^{1/2}$$

Remark: The assumption

$$\int_{\mathbb{R}} \frac{(f_0 - M)^2}{M} \, dv < \infty$$

means that the initial data f_0 is very close to the stationary state M, so it is a very restrictive assumption. For instance, if f_0 is a probability density with polynomial decay strong enough we have

$$\int_{\mathbb{R}} f_0 \log \frac{f_0}{M} dv < \infty, \qquad \int_{\mathbb{R}} \frac{(f_0 - M)^2}{M} \, dv = \infty$$

In the previous analysis we have exploited Log–Sobolev and Poincaré inequalities in order to prove exponential convergence to the Gaussian stationary state of the solution of the Cauchy problem for the Fokker–Planck equation.

We are going to address the following question: **is this strong link between Fokker–Planck equation and functional inequalities specific to the Gaussian density or can it be exploited for other probability densities ?**

General Fokker–Planck equation

Let us consider a general Fokker–Planck equation (FPTT 2017, 2019, 2020, 2022)

$$\partial_t f(v,t) = \partial_v^2 \left(P(v) f(v,t)\right) + \partial_v \left(Q(v) f(v,t)\right), \quad t > 0,\ v \in \mathcal{I} \subseteq \mathbb{R}$$

with P and Q smooth enough, $P \geq 0$.

The diffusion coefficient $P(v)$ is not constant and this is a major difference with the existing litterature.

- The stationary state formally is

$$f_\infty(v) = \frac{C}{P(v)} exp\left(-\int \frac{Q(v)}{P(v)}\, dv\right)$$

- conservation of mass and positivity
- the adjoint equation on $F(t) = f(t)/f_\infty$ is

$$\partial_t F(v,t) = P(v)\partial_v^2 F(v,t) - Q(v)\,\partial_v F(v,t), \quad t > 0,\ v \in \mathcal{I} \subseteq \mathbb{R}$$

Question: can we prove exponential convergence in L^1 to the stationary state in the case of a general Fokker–Planck equation ?

Let us come back to the Shannon entropy of the solution $f(t)$ relative to the stationary state f_∞:

$$H(f(t), f_\infty) = \int_{\mathcal{I}} f(t) \log \frac{f(t)}{f_\infty}\, dv$$

$$= \int_{\mathcal{I}} \sqrt{\frac{f(t)}{f_\infty}}^2 \log \sqrt{\frac{f(t)}{f_\infty}}^2 f_\infty\, dv - \left(\int_{\mathcal{I}} \sqrt{\frac{f(t)}{f_\infty}}^2 f_\infty dv \right) \log \left(\int_{\mathcal{I}} \sqrt{\frac{f(t)}{f_\infty}}^2 f_\infty dv \right)$$

It can be proved (FPTT 2017) that

$$\frac{d}{dt} H(f(t), f_\infty) = -4 \int_{\mathcal{I}} P(v) \left(\partial_v \sqrt{\frac{f(t)}{f_\infty}} \right)^2 f_\infty\, dv$$

(the analogous of Mc Kean result of the decreasing of the relative entropy estimated by a **weighted** Fisher functional)

So, if the following **weighted Log–Sobolev inequality** were true

$$\int_{\mathcal{I}} \varphi^2(v) \log \varphi^2(v)\, f_\infty(v)\, dv - \left(\int_{\mathcal{I}} \varphi^2(v)\, f_\infty(v)\, dv\right) \log \left(\int_{\mathcal{I}} \varphi^2(v)\, f_\infty(v)\, dv\right) \\ \leq C \int_{\mathcal{I}} P(v) (\varphi'(v))^2\, f_\infty(v)\, dv.$$

by applying the inequality to $\varphi = \sqrt{\frac{f(t)}{f_\infty}}$ we could end up with the same conclusion as in the classical Gaussian case

$$\frac{d}{dt} H(f(t), f_\infty) = -4 \int_{\mathcal{I}} P(v) \left(\partial_v \sqrt{\frac{f(t)}{f_\infty}}\right)^2 f_\infty\, dv \leq -CH(f(t), f_\infty)$$

and so

$$H(f(t), f_\infty) \leq e^{-Ct} H(f_0, f_\infty)$$

Analogously, starting from the adjoint equation on $F(t) = f(t)/f_\infty$

$$\partial_t F(v,t) = P(v)\partial_v^2 F(v,t) - Q(v)\,\partial_v F(v,t) \quad t>0,\; v \in \mathcal{I} \subseteq \mathbb{R}$$

and considering the $L^2(f_\infty dv)$ distance of $F(t)$ to 1

$$\int_{\mathcal{I}} (F(t)-1)^2\, f_\infty\; dv = \int_{\mathcal{I}} \left(F(t) - \left(\int_{\mathcal{I}} F(t)\, f_\infty\, dv \right) \right)^2 f_\infty dv$$

it can be proved (FPTT 2017) that

$$\frac{d}{dt}\int_{\mathcal{I}} (F(t)-1)^2\, f_\infty\; dv = -2\int_{\mathcal{I}} P(v)\,(\partial_v F(t))^2\, f_\infty dv$$

If the following **weighted Poincaré inequality** were true

$$\int_{\mathcal{I}} \left(\varphi(v) - \left(\int_{\mathcal{I}} \varphi(v)\, f_\infty(v)\, dv \right) \right)^2 f_\infty(v)\, dv \le C \int_{\mathcal{I}} P(v) (\varphi'(v))^2\, f_\infty(v)\, dv$$

then, letting $\varphi = F(t) = \dfrac{f(t)}{f_\infty}$, we would get as in the classical Gaussian case

$$\frac{d}{dt}\int_{\mathcal{I}} (F(t)-1)^2\, f_\infty\; dv = -2\int_{\mathcal{I}} P(v)\,(\partial_v F(t))^2\, f_\infty dv \le -K\int_{\mathcal{I}} (F(t)-1)^2\, f_\infty\; dv$$

and so

$$\int_{\mathcal{I}} (F(t)-1)^2\, f_\infty\; dv \le e^{-Kt}\int_{\mathcal{I}} \left(\frac{f_0}{f_\infty} - 1 \right)^2 f_\infty\; dv$$

Both inequalities

$$H(f(t), f_\infty) \leq e^{-Ct} H(f_0, f_\infty) \tag{1}$$

and

$$\int_{\mathcal{I}} (F(t) - 1)^2 f_\infty \, dv \leq e^{-Kt} \int_{\mathcal{I}} \left(\frac{f_0}{f_\infty} - 1 \right)^2 f_\infty \, dv \tag{2}$$

would imply exponential convergence in L^1 of the solution to the stationary state, but the assumption on the initial data is stronger in the case of application of Poincaré inequality (2) than in the case of Log–Sobolev inequality (1) since for f_∞ bounded and with polynomial decay at infinity it can be proved that

$$\int_{\mathcal{I}} f_0 \log \frac{f_0}{f_\infty} \, dv \leq C \int_{\mathcal{I}} \frac{(f_0 - f_\infty)^2}{f_\infty} \, dv$$

We are going to focus on the **inverse–Gamma densities** [FPTT17], FPTT20, FPTT22]

$$f_\infty(v) = C_{m,\beta} \frac{\exp\left(-\frac{m}{v}\right)}{v^{2\beta}}, \quad v \in \mathbb{R}^+$$

where $m > 0$, $\beta > \frac{1}{2}$ and the constant $C_{m,\beta}$ is chosen to fix the total mass of f_∞ equal to one.
Each of these densities was considered in 2005 by Pareschi and Toscani as a stationary state of a Fokker–Planck equation describing the distribution of the wealth in occidental countries. They exhibit a power-law tail for large values of the wealth variable (so–called **heavy tailed** densities).

Questions:

- can we prove a (possibly weighted) Log–Sobolev inequality or a (possibly weighted) Poincaré inequality satisfied by an inverse–Gamma density ?
- Which weight has to be chosen ?
- Are these (possibly weighted) inequalities suitable for proving exponential convergence in L^1 for the solutions of a Fokker–Planck equation which has f_∞ as stationary state?

Functional inequalities for general densities have a long history (Brascamp–Lieb 76, Bakry–Emery 85, Klaassen 85, Beckner 98, Markowich–Villani 2000, Arnold–Markowich–Toscani–Unterreiter 2001, Bobkov–Ledoux 2009, Bonnefont–Joulin–Ma 2014, 2016, . . .)

We would like to underline here the deep relation between functional inequalities and the Fokker–Planck equations which have these densities as a stationary state.

The inverse–Gamma density $f_\infty(v) = C_{m,\beta} \dfrac{\exp\left(-\frac{m}{v}\right)}{v^{2\beta}}$ is stationary state of the original model introduced by Pareschi and Toscani

$$\partial_t f(v,t) = \partial_v^2 \left(v^2 f(v,t) \right) + (2\beta - 2)\partial_v \left(\left(v - \frac{m}{2\beta - 2} \right) f(v,t) \right)$$

but also of a whole family of equations

$$\partial_t f(v,t) = \partial_v^2 \left(v^{2\alpha} f(v,t) \right) + \lambda \partial_v \left(\left(v - \frac{m}{\lambda} \right) v^{2\alpha - 2} f(v,t) \right)$$

for

$$2\alpha + \lambda = 2\beta$$

so that we can consider all $2\beta > 1$ for suitable λ and α under some constraints that will be made precise in the sequel.

Remark: For these equations the well–posedness of a Cauchy problem with L^1 initial data follows by a result by Feller 52 which makes all the computations fully justified.

Weighted Poincaré inequality

Theorem (FPTT 2017, FPTT 2022)

Let $f_\infty = C_{m,\beta} \dfrac{\exp\left(-\frac{m}{v}\right)}{v^{2\beta}}$ *an inverse–Gamma density, for* $v \in \mathbb{R}_+$, $\beta > 1/2$, $m > 0$. *For any smooth function* φ *on* $\mathbb{R}_+$ *so that*

$$\int_0^{+\infty} \left(\varphi(v) - \left(\int_0^{+\infty} \varphi(v) f_\infty(v) dv\right)\right)^2 f_\infty(v) dv < \infty$$

it holds

$$\int_0^{+\infty} \left(\varphi(v) - \left(\int_0^{+\infty} \varphi(v) f_\infty(v) dv\right)\right)^2 f_\infty(v) dv \le \frac{1}{\rho(\beta)} \int_0^{+\infty} v^2 (\varphi'(v))^2 f_\infty(v) dv$$

where

$$\rho(\beta) = \begin{cases} \left(\beta - \frac{1}{2}\right)^2 & \frac{1}{2} < \beta \le \frac{3}{2} \\ 2(\beta - 1) & \beta > \frac{3}{2}. \end{cases}$$

Poincaré inequality $\Rightarrow$ exponential convergence in L^1 with strong assumption on the initial data

For the **original wealth model** (Pareschi Toscani 2005)

$$\partial_t f(v,t) = \partial_v^2 \left(v^2 f(v,t) \right) + (2\beta - 2)\partial_v \left(\left(v - \frac{m}{2\beta - 2} \right) f(v,t) \right)$$

the previous result

$$\int_0^{+\infty} \left(\varphi(v) - \left(\int_0^{+\infty} \varphi(v) f_\infty(v) dv \right) \right)^2 f_\infty(v) dv \leq \frac{1}{\rho(\beta)} \int_0^{+\infty} v^2 (\varphi'(v))^2 f_\infty(v) dv$$

implies exponential L^1 convergence for the solutions with initial data $f_0 \in L^1$ satisfying the strong assumption

$$\int_0^{+\infty} \frac{(f_0 - f_\infty)^2}{f_\infty} dv < \infty$$

to the Gamma–inverse stationary state f_∞.

Weighted Log–Sobolev inequality

Theorem (FPTT 2020, 2022)

Let $f_\infty = C_{m,\beta} \dfrac{\exp\left(-\frac{m}{v}\right)}{v^{2\beta}}$ *an inverse–Gamma density, for* $v \in \mathbb{R}_+$, $\beta > 1/2$, $m > 0$. *For any bounded smooth function* φ *on* $\mathbb{R}_+$ *such that*

$$\int_0^{+\infty} \varphi(v) \log \frac{\varphi(v)}{f_\infty(v)} dv < \infty$$

and for all $1 < \alpha \leq \frac{3}{2}$ *and* $\alpha < \beta + \frac{1}{2}$ *one has the bound*

$$\int_0^{+\infty} \varphi(v) \log \frac{\varphi(v)}{f_\infty(v)} dv \leq \gamma_{\beta,\alpha,m} \int_0^{+\infty} v^{2\alpha} (\varphi'(v))^2 f_\infty(v) dv$$

where $\gamma_{\beta,\alpha,m} > 0$ *is an explicit constant.*

Remark: The inequality

$$\int_0^{+\infty} \varphi(v) \log \frac{\varphi(v)}{f_\infty(v)} dv \leq \gamma_{\beta,m} \int_0^{+\infty} v^{2} (\varphi'(v))^2 f_\infty(v) dv$$

does not seem to hold (as we will see in a while).

Log–Sobolev inequality $\Rightarrow$ exponential convergence in L^1 with mild assumption on the initial data

For a **modified wealth model** (FPTT 2020)

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial^2}{\partial v^2}\left(v^{2\alpha} f(v,t)\right) + \lambda \frac{\partial}{\partial v}\left(\left(v - \frac{m}{\lambda}\right) v^{2\alpha-2} f(v,t)\right)$$

the previous result

$$\int_0^{+\infty} \varphi(v) \log \frac{\varphi(v)}{f_\infty(v)} dv \le \gamma_{\beta,\alpha,m} \int_0^{+\infty} v^{2\alpha} (\varphi'(v))^2 f_\infty(v) dv$$

implies exponential L^1 convergence to the Gamma–inverse stationary state f_∞ for the solutions with initial data $f_0 \in L^1$ satisfying the mild assumption

$$\int_0^{+\infty} f_0(v) \log \frac{f_0(v)}{f_\infty(v)} dv < \infty$$

All the proofs exploit in an essential way the link between the probability density f_∞ and the Fokker–Planck equation which has f_∞ as a stationary state. This is the major contribution of our work. The balance between the diffusion coefficient $P(v)$ and the drift coefficient $Q(v)$ is essential for proving Poincaré vs Log–Sobolev inequalities. On the other hand, the inequalities allow us to prove convergence of the solutions of the Fokker–Planck equation to the stationary state f_∞.

Further results

1. Optimal weight growth for Log–Sobolev inequality.

Theorem (FPTT 2022)

Let $f_\infty = C_{m,\beta} \dfrac{\exp\left(-\frac{m}{v}\right)}{v^{2\beta}}$ *an inverse–Gamma density, for* $v \in \mathbb{R}_+$, $\beta > 1/2$, $m > 0$. *For any smooth function* φ *on* $\mathbb{R}_+$ *such that*

$$\int_0^{+\infty} \varphi(v) \log \frac{\varphi(v)}{f_\infty(v)} dv < +\infty$$

one has the bound

$$\int_0^{+\infty} \varphi(v) \log \frac{\varphi(v)}{f_\infty(v)} dv \le \frac{2}{\rho_{\beta,m}} \int_0^{+\infty} v^2 \log(1+v)(\varphi'(v))^2 f_\infty(v) dv,$$

where

$$\rho_{\beta,m} = \min\left\{\frac{m}{2}, (\beta - 1)\right\}.$$

2. Weighted Log–Sobolev inequality for other probability densities f_∞ on $I \subseteq \mathbb{R}$:

- Cauchy–type [Bobkov–Ledoux 2009, Saumard 2019, FPTT 2022]

$$f_\infty(v) = C_\beta \frac{1}{(1+v^2)^\beta}, \quad v \in \mathbb{R}, \beta > \frac{1}{2}$$

- Beta-type [Epstein–Mazzeo 2010, FPTT 2019]

$$f_\infty(v) = c_m \left(\frac{1}{1+v}\right)^{1-(1+m)/2} \left(\frac{1}{1-v}\right)^{1-(1-m)/2}, \quad -1 < m < 1, \quad v \in (-1,1)$$

3. Weighted Poincaré and Wirtinger inequalities for general probability densities f_∞ on $I \subseteq \mathbb{R}$ [FPTT2022]: for $1 \le p < +\infty$

$$\int_{\mathcal{I}} \left| \varphi(v) - \left(\int_{\mathcal{I}} \varphi(v)\, f_\infty(v)\, dv \right) \right|^p f_\infty(v)\, dv \le C \int_{\mathcal{I}} w(v) |\varphi'(v)|^p f_\infty(v)\, dv$$

Poincaré inequalities for radially symmetric densities on $\mathbb{R}^n$

Let us consider $f_\infty(v) = f_\infty(|v|)$, $v \in \mathbb{R}^n$ a probability density. We will consider as an example f_∞ a generalized Cauchy density

$$f_\infty(v) = C_\beta \frac{1}{(1+|v|^2)^\beta}, \quad v \in \mathbb{R}^n, \beta > \frac{n}{2}$$

Theorem (Bobkov-Ledoux, 2009)

Let $f_\infty(v) = C_\beta \frac{1}{(1+|v|^2)^\beta}$ *a generalized Cauchy density, for* $v \in \mathbb{R}^n$, $\beta \geq n$. *For any bounded smooth function* φ *on* $\mathbb{R}^n$ *such that variance of* φ *with respect to* f_∞ *is finite, the following Poincaré inequality is true*

$$\int_{\mathbb{R}^n} \left(\varphi(v) - \left(\int_{\mathbb{R}^n} \varphi(v) f_\infty(v)\, dv\right)\right)^2 f_\infty(v)\, dv \leq \frac{C}{\beta} \int_{\mathbb{R}^n} \left(1+|v|^2\right) |\nabla\varphi(v)|^2 f_\infty(v)\, dv$$

where $C > 0$ *is a constant.*

We proved a similar result for the generalized Cauchy densities, for a slightly wider range of parameters, using

- spherical coordinates
- a Poincaré inequality for n–dimensional probability densities which are a product of 1–dimensional probability densities.

Theorem (FPTT 2023)

Let $f_\infty(v) = C_\beta \dfrac{1}{(1+|v|^2)^\beta}$ *a generalized Cauchy density, for* $v \in \mathbb{R}^n$, $\beta > \frac{n+1}{2}$. *For any bounded smooth function* φ *on* $\mathbb{R}^n$ *such that the variance of* φ *with respect to* f_∞ *is finite, the following Poincaré inequality is true*

$$\int_{\mathbb{R}^n} \left(\varphi(v) - \left(\int_{\mathbb{R}^n} \varphi(v) f_\infty(v)\, dv\right)\right)^2 f_\infty(v)\, dv \le C(\beta) \int_{\mathbb{R}^n} \left(1 + |v|^2\right) |\nabla\varphi(v)|^2 f_\infty(v)\, dv$$

where $C(\beta) > 0$ *is an explicit constant which blows up as* $\beta \to \frac{n+1}{2}$.

Idea of the proof.

Let us consider $n = 3$. Since f_∞ is radially symmetric and so $f_\infty(v) = f_\infty(|v|)$, we write in spherical coordinates

$$\int_{\mathbb{R}^3} \left(\varphi(v) - \left(\int_{\mathbb{R}^3} \varphi(v) f_\infty(v)\, dv\right)\right)^2 f_\infty(v)\, dv$$

$$= \int_{\mathbb{R}^3} \varphi^2(v) f_\infty(v)\, dv - \left(\int_{\mathbb{R}^3} \varphi(v) f_\infty(v)\, dv\right)^2$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \varphi^2(\rho, \theta_1, \theta_2) f_\infty(\rho) \rho^2 \sin\theta_1 \, d\rho\, d\theta_1\, d\theta_2$$

$$- \left(\int_0^\infty \int_0^\pi \int_0^{2\pi} \varphi(\rho, \theta_1, \theta_2) f_\infty(\rho) \rho^2 \sin\theta_1 \, d\rho\, d\theta_1\, d\theta_2\right)^2$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \varphi^2(\rho, \theta_1, \theta_2) 4\pi\rho^2 f_\infty(\rho) \frac{\sin\theta_1}{2} \frac{1}{2\pi} \, d\rho\, d\theta_1\, d\theta_2$$

$$- \left(\int_0^\infty \int_0^\pi \int_0^{2\pi} \varphi(\rho, \theta_1, \theta_2) 4\pi\rho^2 f_\infty(\rho) \frac{\sin\theta_1}{2} \frac{1}{2\pi} \, d\rho\, d\theta_1\, d\theta_2\right)^2,$$

- $4\pi\rho^2 f_\infty(\rho)$ is a density on $(0, \infty)$
- $\frac{\sin\theta_1}{2}$ is a density on $(0, \pi)$
- $\frac{1}{2\pi}$ is a density on $(0, 2\pi)$

Where is Fokker–Planck?

If $f : (a, b) \to \mathbb{R}, (a, b) \subset \mathbb{R}$ is probability density satisfying

$$\int_a^b vf(v)dv = m \in \mathbb{R}$$

then f solves the stationary Fokker–Planck equation

$$\partial_v(K(v)f(v)) + (v - m)f(v) = 0$$

where

$$K(v) = \begin{cases} \dfrac{\int_a^v (m-y)f(y)dy}{f(v)}, & a < v \le m \\ \dfrac{\int_v^b (y-m)f(y)dy}{f(v)}, & m < v < b. \end{cases}$$

Remark: it is the same equation as for the Maxwellian density, for which $m = 0$ and $K(v) = 1$

$$\partial_v M(v) + vM(v) = 0$$

Theorem (FPTT 2023)

Let us consider a probability density on $A_n \subseteq \mathbb{R}^n$, $f(v) = f(v_1, \ldots, v_n)$ *defined as a product of one-dimensional functions:*

$$f(v) = \prod_{i=1}^{n} f_i(v_i),$$

where each $f_i(v_i)$ *is a probability density on* $(a_i, b_i) \subset \mathbb{R}$, *such that*

$$\int_{a_i}^{b_i} v_i f_i(v_i)\, dv_i = m_i \in \mathbb{R}.$$

Let us denote by A_n *the set in* $\mathbb{R}^n$ *defined by* $A_n = \prod_{i=1}^{n} (a_i, b_i)$.

For any absolutely continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, *such that the variance of* φ *with respect to* f *is finite, it holds*

$$\int_{\mathbb{R}^n} \left(\varphi(v) - \left(\int_{\mathbb{R}^n} \varphi(v) f(v)\, dv \right) \right)^2 f(v)\, dv \le \sum_{i=1}^{n} \int_{A_n} K_i(v_i) \left[\frac{\partial \varphi(v)}{\partial v_i} \right]^2 f(v) dv$$

where $K_i(v)$ *are defined as in the previous slide.*

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Thank you for your attention !