The problem of regularization by noise of the 3D Navier-Stokes equation

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Conference in honor of Pierre Gilles Lemarié-Rieusset

Paris 2024

History

- Attempts on 3D NS with additive noise
  - (1995-2010)
- Transport noise on simpler equations
  - (2010-2018)
- Transport noise for 3D NS
  - (2018-2024)

The 3D Navier-Stokes equations

Space domain: $\mathbb{T}^3$ (or more realistic)
$u = u(x, t)$ velocity field
$p = p(x, t)$ pressure field
$\nu > 0$ viscosity

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u \\ \operatorname{div} u &= 0 \\ u\big|_{t=0} &= u_0 \end{aligned}$$

plus periodic (or more realistic) boundary conditions.

The 3D Navier-Stokes equations

H = closure in $L^2(\mathbb{T}^3, \mathbb{R}^3)$ of smooth fields s.t. $\mathrm{div} = 0$ and $\int = 0$
V = closure in $W^{1,2}(\mathbb{T}^3, \mathbb{R}^3)$ as above

Theorem

i) $u_0 \in H \Longrightarrow \exists u \in C_w([0,T]; H) \cap L^2(0,T;V)$
ii) $u_0 \in V \Longrightarrow \exists \tau > 0, \exists! u \in C(0,\tau;V)$ *(and more regular)*
iv) partial regularity theory; $d_H(\mathit{sing.\ set}) \leq 1$
v) geometric criteria, e.g. on direction of vorticity

Open : $\tau = +\infty$

Smoothing effect of disorder

- Joseph Boussinesq, 1877: *"Turbulent fluctuations are dissipative on the mean flow"*
- John D. Gibbon, '20: ... This runs counter to the traditionally held theory of viscous turbulence in which singularities are related to turbulent dynamics. Adjustment to find the smoothest, most dissipative set could be a way of the flow re-organizing & regularizing itself to avoid singularities.
- Easier (but potentially related) question: could noise help?

Could noise restore uniqueness for the 3D NS equations?

Certainly starting around 1995, but maybe before, some people dreamed to prove that, for the stochastic equation

$$\begin{aligned} du + \left(u \cdot \nabla u + \nabla p - \nu \Delta u\right) dt &= Q^{1/2} dW \\ \operatorname{div} u &= 0 \\ u|_{t=0} &= u_0 \end{aligned}$$

some kind of uniqueness result could hold, for weak solutions.

Where does this hope come from?

Theorem (Veretennikov '81)

If $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable bounded and $(W_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}^d$, for every $x_0 \in \mathbb{R}^d$ there is a unique solution of the equation

$$X_t = x_0 + \int_0^t b(X_s, s)\, ds + W_t.$$

- Many variants and generalizations, also to abstract equations in infinite dimensions (Da Prato, Roeckner, Priola, Veretennikov, Menozzi, Chaudru de Raynal, and many others) but not direct generalizations to fluid dynamic equations until now.
- However, most convex integration negative results about uniqueness have been extended to the stochastic case (hence maybe there is no hope), by Hofmanova and coworkers.

Kolmogorov and uniqueness

Given an SDE, $X_t \in H$

$$dX_t = b(X_t)\,dt + Q^{1/2} dW_t, \qquad X_0 = x_0$$

if the backward Kolmogorov equation, $U = U(x,t)$, $x \in H$, $t \in [0,T]$

$$\partial_t U + \frac{1}{2} Tr\left(QD^2 U\right) + \langle b(x), DU \rangle = 0, \qquad U|_{t=T} = \phi$$

has suitably regular solutions ($\sim$ bounded DU), then

$$\mathbb{E}\left[\phi(X_T)\right] = U(x_0, 0).$$

This is the basis for a uniqueness-in-law result. Pathwise uniqueness under additional regularity of U.

The intuition

The intuition behind the uniqueness result for

$$X_t = x_0 + \int_0^t b(X_s, s)\, ds + W_t$$

is that

- b has certain "singularities", a certain (not too large) singular set
- and W fluctuates so strongly and regularly, to avoid (or average) them.
- (What about the singular set in H of NS...)

The 3D Navier-Stokes equations

Theorem (Da Prato-Debussche JMPA'03)

The infinite dimensional Kolmogorov equation associated to the stochastic 3D Navier-Stokes equations has unique smooth solutions $U(t, u)$.

Problem

However, essentially, $u \in V$ (admissible only locally in time).

Theorem (F.-Romito PTRF'08)

If Q is sufficiently non degenerate, all Markov selections are Strong Feller in $u_0 \in V$, globally in time.

Strong Feller means H-continuous dependence in law, on $u_0 \in V$, globally in time. In a sense, there exists a continuous flow.

Could noise prevent blow-up of the 3D NS equations?

- The hope that additive noise may restore the uniqueness of weak solutions of the 3D Navier-Stokes equations was a bit abandoned around 2010 (perhaps with the exception of Giuseppe Da Prato).
- The later results of convex integration stabilized this decline.
- *Remark: Transport noise restores uniqueness of 2D Euler with L^2 vorticity: Coghi-Maurelli 2023.*
- The alternative, a posteriori more fruitful, idea is that **maybe noise prevents blow-up**.
- The intuition is that blow-up requires a certain degree of organization (e.g. some form of self-similarity) and random kicks (even endogenously produced by turbulence) may disrupt "well prepared" paths to blow-up.

Could noise prevent blow-up of the 3D NS equations?

A Leray solution (which exists) of

$$\begin{aligned} du + (u \cdot \nabla u + \nabla p - \nu \Delta u)\, dt &= Q^{1/2} dW \\ \operatorname{div} u &= 0 \\ u|_{t=0} &= u_0 \end{aligned}$$

is a stochastic process ($(\Omega, F, \mathbb{P})$ being a probability space)

$$u : \Omega \to C_w ([0, T]; H) \cap L^2 (0, T; V)$$

with suitable progressive measurability properties, satisfying

$$\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] + 2\nu \int_0^t \mathbb{E}\left[\|\nabla u(s)\|_{L^2}^2\right] ds \leq \mathbb{E}\left[\|u(0)\|_{L^2}^2\right] + Tr(Q).$$

First remarks on stochastic 3D Navier-Stokes equations

We say that

$$u : \Omega \longrightarrow C_w\left([0,\infty); H\right) \cap L^2_{loc}\left(0,\infty; V\right)$$

is time-stationary if its law is invariant by time-translations, or equivalently if

$$\mathbb{E}\left[\phi\left(u\left(t_1+h\right),...,u\left(t_n+h\right)\right)\right]$$

is independent of h for every choice of n, $t_1,...,t_n$ and $\phi : H^n \longrightarrow \mathbb{R}$.
If u is time-stationary with $u\left(0\right)$ square integrable in $W^{1,2}$, then

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left\|\nabla u\left(t\right)\right\|^2_{L^2}\right] = \mathbb{E}\left[\left\|\nabla u\left(0\right)\right\|^2_{L^2}\right] < \infty.$$

This hints at special regularity properties: if we could put $\sup_{t\in[0,T]}$ inside expectation, then solutions would be globally smooth (and unique).

Local regularity theory

At the beginning of 2000, my hope was that a stochastic analog of Caffarelli-Kohn-Nirenberg partial regularity theory could give deeper information.

$$\begin{aligned} du + (u \cdot \nabla u + \nabla p - \nu \Delta u)\, dt &= Q^{1/2} dW \\ \operatorname{div} u &= 0 \qquad u|_{t=0} = u_0 \end{aligned}$$

Theorem (F.-Romito TAMS'02)

There exist weak solutions satisfying the local energy inequality. Moreover, if

$$\limsup_{r \to 0} \frac{1}{r} \int\int_{Q_r(x_0,t_0)} |\nabla u|^2 \, dxdt$$

is small enough, then (x_0, t_0) is a regular point ($Q_r(x_0,t_0) = B(x_0,r) \times [t_0 - r^2, t_0]$).

Local regularity theory for stationary solutions?

In the deterministic case, if u is a stationary (hence *constant-in-time*) solution, then

$$\frac{1}{r}\int\int_{Q_r(x_0,t_0)} |\nabla u|^2 \, dxdt \leq r \int_{\mathbb{T}^3} |\nabla u|^2 \, dx$$

hence there are no singular points.
What happens in the stochastic case to stationary solutions?

Let $\Upsilon \subset [0,\infty)$ be a Borel set. Call $\mathcal{R}_{\mathbb{T}^3\times\Upsilon}(\omega)$ the set of regular points in $\mathbb{T}^3 \times \Upsilon$, depending on the random parameter $\omega \in \Omega$.

Theorem (F.-Romito TAMS'02)

For time-stationary solutions with $u(0)$ square integrable in $W^{1,2}$, for every $t_0 \geq 0$, $\mathbb{T}^3 \times \{t_0\}$ is regular for a.e. path, namely

$$\mathbb{P}\left(\omega \in \Omega : \mathcal{R}_{\mathbb{T}^3\times\{t_0\}}(\omega) = \mathbb{T}^3 \times \{t_0\}\right) = 1.$$

Heuristically, singularities are not visible in $\mathbb{T}^3 \times \{t_0\}$ (the event that a singularity appears in $\mathbb{T}^3 \times \{t_0\}$ has probability zero).
This is a result of "regularization by stationarity" (not regularization by noise). Indeed, it is true also for stationary solutions of the deterministic equation.
It *should* be true also for sets Υ of Hausdorff dimension $< \frac{1}{2}$.

Stationary solutions are quite representative

- As said, the result is true also without noise.
- But with (sufficiently non-degenerate) noise, stationary solutions "experience" the full space H.
- By disintegration with respect to initial condition, one can deduce a result for individual solutions (not stationary) starting from a large set of initial conditions.
- With more effort, from every initial condition it *should* be possible to get the same result (by a Strong-Feller argument).

About the proof

The idea of proof is relatively simple. Recall the criterium: if

$$\limsup_{r\to 0}\frac{1}{r}\int\int_{Q_r(x_0,t_0)}|\nabla u|^2\,dxdt=0$$

then (x_0,t_0) is a regular point. For stationary solutions $\mathbb{E}\left[\|\nabla u(t)\|_{L^2}^2\right]=: C<\infty$, hence (like determ. case)

$$\mathbb{E}\left[\frac{1}{r}\int\int_{\mathbb{T}^3\times[t_0-r^2,t_0]}|\nabla u|^2\,dxdt\right]\leq Cr.$$

By a Borel-Cantelli argument

$$\frac{1}{r}\int\int_{\mathbb{T}^3\times[t_0-r^2,t_0]}|\nabla u|^2\,dxdt$$

is a.s. small, hence also $\frac{1}{r}\int\int_{Q_r(x_0,t_0)}|\nabla u|^2\,dxdt$ for every $x_0\in\mathbb{T}^3$.

The Borel-Cantelli argument

Choose (r_n) decreasing to zero; set

$$A_n = \left\{ \omega \in \Omega : \frac{1}{r_n} \int\int_{\mathbb{T}^3 \times [t_0 - r_n^2, t_0]} |\nabla u|^2 \, dxdt \geq r_n^{1/2} \right\}.$$

Then (Markov inequality)

$$\mathbb{P}(A_n) \leq \frac{1}{r_n^{1/2}} \mathbb{E} \left[\frac{1}{r_n} \int\int_{\mathbb{T}^3 \times [t_0 - r_n^2, t_0]} |\nabla u|^2 \, dxdt \right] \leq C r_n^{1/2}$$

Then, if $\sum_n r_n^{1/2} < \infty$, we have $\sum_n \mathbb{P}(A_n) < \infty$ (hypothesis of Borel-Cantelli lemma), therefore a.e. ω belongs to A_n^c for n large enough, namely a.s.

$$\frac{1}{r_n} \int\int_{\mathbb{T}^3 \times [t_0 - r_n^2, t_0]} |\nabla u|^2 \, dxdt < r_n^{1/2}$$

for n large enough.

A solution on $\mathbb{R}^3$ (or $\mathbb{T}^3$) is space homogeneous if, for every $t \in [0, T]$, the law of $x \mapsto u(x, t)$ is invariant by space-translations, namely

$$\mathbb{E}\left[\phi\left(u\left(x_1 + h, t\right), ..., u\left(x_n + h, t\right)\right)\right]$$

is independent of h, for every choice of t, n, $x_1, ..., x_n$ and $\phi : \mathbb{R}^{3n} \longrightarrow \mathbb{R}$.

Theorem (A. Basson CMP '06, SPA '08)

There exist space stationary solutions, satisfying the local energy inequality.

Hence satisfying CKN theory.

Given $D \subset \mathbb{R}^3$, call

$$\mathcal{R}_{D\times[0,T]}(\omega) = \{(x,t) : x \in D, t \in [0,T], (x,t) \text{ regular}\}.$$

The following result *should* be true.

Theorem (only conjectured)

For space homogeneous solutions, if D has Hausdorff dimension < 2, then $D \times [0,T]$ is regular for a.e. path, namely

$$\mathbb{P}\left(\omega \in \Omega : \mathcal{R}_{D\times[0,T]}(\omega) = D \times [0,T]\right) = 1.$$

Idea of proof: $D \subset \cup_i B_{r_i}\left(x_0^i\right)$, $\sum_i r_i^2$ infinitesimal

$$\mathbb{E}\left[\frac{1}{r}\int\int_{\left(\cup_i B_{r_i}\left(x_0^i\right)\right)\times[0,T]} |\nabla u|^2 \, dxdt\right] \leq C\frac{\sum_i r_i^3}{r} \text{ small}$$

Then Borel-Cantelli.

Is there a future for regularization by noise and local regularity theory?

- My conjecture is yes, but it looks very difficult.
- One should perhaps exploit the transport-stretching properties of noise (= small-scale perturbations).
- Stochastic transport has a smoothing effect, by mixing, similar to diffusion
- Stochastic stretching is the hard part of the question.
  - Clearly it may produce an average increase of lengths
  - But maybe stochastically stabilized.

Transport noise on linear equations

Theorem (F.-Gubinelli-Priola, IM '10)

If $b \in C_b^{\alpha}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ with $\operatorname{div} b$ bounded, then the linear stochastic transport equation

$$du + b \cdot \nabla u dt = \nabla u \circ dW$$

is well posed in $L^{\infty}\left(\mathbb{R}^d\right)$.

Notice that classical Di Perna - Lions theory, without noise, requires $b \in W^{1,1}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and the best possible generalizations (e.g. Ambrosio) reach BV class.

The stochastic result was extended to bounded measurable drift and beyond, by several authors.

The reason and the limitation

The stochastic characteristics behind

$$du + b \cdot \nabla u dt = \nabla u \circ dW$$

are precisely

$$dX_t = b(X_t)\, dt + Q^{1/2} dW_t, \qquad X_0 = x_0$$

One can improve Veretennikov '81 result and prove existence of a relatively smooth stochastic flow $\varphi_t(x)$
Then by suitably generalized commutator estimates one can prove

$$u(\varphi_t(x), t) = u_0(x).$$

It does't work for nonlinear problems (e.g. because b is random and could "adapt" its singularities to W; think to $b(x,t) = \sqrt{x - W_t}$).

Rich transport noise on nonlinear equations

Theorem (F.-Gubinelli-Priola, SPA '11)

If $\eta = \sum_k \sigma_k(x) \frac{dW_t^k}{dt}$ *with suitable Hormander conditions on* (σ_k)*, the point vortex solutions* $\xi(t, \cdot) = \sum_{i=1}^N \alpha_i \delta_{X_t^i}$ *of the stochastic 2D Euler equation*

$$\partial_t \xi + u \cdot \nabla \xi = \eta \circ \nabla \xi$$

do not have collisions (hence are well posed) for every initial configuration.

- Generalized and refined for point-charge solutions of 1D Vlasov-Poisson equation, Delarue-F.-Vincenzi CPAM'14.
- In these simple examples one can also address the question of continuation (by zero-noise limit) after the singularity.

The principle used here

The point vortex equation

$$dX_t^i = \sum_{j \neq i} \left(\frac{\Gamma j}{2\pi} \frac{\left(X_t^i - X_t^j\right)^\perp}{\left|X_t^i - X_t^j\right|^2} + \gamma\left(X_t^i, X_t^j\right) \right) dt + \eta\left(X_t^i, dt\right)$$

leaves Lebesgue measure in $\mathbb{T}^{2N}$ invariant and is well posed for a.e. initial condition

$$\left(X_0^1, ..., X_0^N\right) \in \mathbb{T}^{2N}$$

(similarly to several equations of dispersive type).
Then, thanks to a rich noise, *every* initial condition immediately enters the good full-measure set.

Breakthrough on transport noise

Theorem (Galeati SPDE'20)

Under suitable assumptions on σ_k^ϵ, *the solution* θ_ϵ *of*

$$\begin{aligned} \partial_t \theta_\epsilon + \eta_\epsilon \circ \nabla \theta_\epsilon &= 0 \\ \theta_\epsilon|_{t=0} &= \theta_0 \end{aligned} \qquad \eta_\epsilon = \sum_k \sigma_k^\epsilon (x) \frac{dW_t^k}{dt}$$

converges (in a suitable sense) to the solution of the deterministic equation

$$\begin{aligned} \partial_t \Theta &= \mathcal{L}\Theta \\ \Theta|_{t=0} &= \theta_0 \end{aligned}$$

where

$$(\mathcal{L}\theta)(x) = \frac{1}{2} \operatorname{div} (Q(x) \nabla \theta(x)).$$

Consequences on Regularization by Noise

One can transfer estimates from the limit "smooth" PDE to the approximating SPDE, proving delay of blow-up of regular solutions:

- F-Galeati-Luo, CPDE'21: Keller-Siegel and other nonlinear models
- Agresti, '22: systems of reaction diffusion equations
- F-Luo, PTRF'21 and F-Hofmanova-Luo-Nilssen, AAP'22, for 3D Navier-Stokes equations (but only transport noise, not advection)

Back to the 3D Navier-Stokes equations

Let me describe the problem and the result for the Navier-Stokes equation in vorticity form

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \nu \Delta \omega$$

where $\omega = \operatorname{curl} u =$ vorticity. Its stochastic counterpart must be

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u + \boxed{\eta \circ \nabla \omega \text{-} \omega \circ \nabla \eta} = \nu \Delta \omega$$

with η white noise perturbation of u.
The question is whether it converges to

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \nu \Delta \omega + \boxed{\operatorname{div}\left(Q \nabla \omega\right)}$$

3D Navier-Stokes equations with only transport noise

Instead of the equation with *advection* noise

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u + \boxed{\eta \circ \nabla \omega \text{-} \omega \circ \nabla \eta} = \nu \Delta \omega$$

we are able to deal only with *transport* noise

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u + \boxed{\eta \circ \nabla \omega} = \nu \Delta \omega$$

because the random stretching term

$$\omega \circ \nabla \eta$$

has uncontrollable effects.

Only transport noise, but projected

The equation

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u + \boxed{\eta \circ \nabla \omega} = \nu \Delta \omega$$

however is meaningless since $\operatorname{div}(LHS) = 0$, but $\operatorname{div}(\eta \circ \nabla \omega)$ is not necessarily zero.

Thus, a meaningful equation with only transport noise is

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u + \boxed{\mathrm{P}(\eta \circ \nabla \omega)} = \nu \Delta \omega$$

where P is the projection on divergence free fields

$$\mathrm{P} : L^2\left(\mathbb{T}^3, \mathbb{R}^3\right) \to H$$

The result

Consider

$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla \omega_\epsilon - \omega_\epsilon \cdot \nabla u_\epsilon + \boxed{\mathsf{P}(\eta_\epsilon \circ \nabla \omega)} = \nu \Delta \omega_\epsilon$$

Theorem (Flandoli-Luo, PTRF '21)

i) Given $\omega_0 \in H$, In a suitable scaling limit of η_ϵ the maximal solution ω_ϵ is close to the maximal solution of

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = (\nu + K)\, \Delta \omega.$$

ii) Up to a small probability, there exists ϵ such that maximal time of ω_ϵ is arbitrarily large.

A recent manuscript

Theorem (A. Agresti '24)

Consider the hyper dissipative NS equation with transport noise:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p + \nu (-\Delta)^{1+\epsilon} u &= (\eta \circ \nabla u) \\ \operatorname{div} u &= 0 \\ u|_{t=0} &= u_0 \end{aligned}$$

For a suitable choice of noise, blow-up is prevented.

Summary and remarks

- Regularizing smooth solutions (in the sense of blow-up) looks more promising than uniqueness by noise of weak solutions.
- Stochastic stretching is the bottleneck. Moreover, large noise (to have large turbulent diffusion) is not so realistic. The right picture could be small turbulent diffusion and large random stretching.
- Recent works on stochastic stretching for easier problems (passive magnetic field, passive polymers) looking for technical ideas to attach vorticity stretching
    - random stretching increases the length of vector fields, but also spreads in a regular statistical way their length.

Recall:

- Joseph Boussinesq, 1877: "Turbulent fluctuations are dissipative on the mean flow"
- John D. Gibbon, '20: ... This runs counter to the traditionally held theory of viscous turbulence in which singularities are related to turbulent dynamics. Adjustment to find the smoothest, most dissipative set could be a way of the flow re-organizing & regularizing itself to avoid singularities.

We have reached some additional understanding of the dissipation properties but they are not sufficient, the effect of random stretching looks more important.
In particular, could random stretching mitigate an exploding deterministic stretching?

Thank you!

and greetings to Pierre Gilles!